Rep Theory Talk: Bradley, Nov. 3

Outline:

$$\begin{tikzcd}
\dot{\mathcal{U}}_q(sl_2) \arrow[d, squiggly, "\text{Nick}"'] & & & \text{Khovanov Homology} \\
U_q(sl_2) \arrow[r, squiggly, "\text{Me (today)}"] \arrow[d, "\text{is a...}"'] & \text{Kauffman bracket} \arrow[rr, "\text{slight modification}"] \arrow[d, "\text{is a...}"] & & \text{Jones polynomial} \arrow[u, squiggly, "\text{Me (Nov. 21)}"'] \\
\text{Ribbon Hopf Algebra} \arrow[r, "\text{gives us...}"] & \text{Operator invariant of framed oriented tangles} & &
\end{tikzcd}$$

Hopf Algebras

Defn: A Hopf algebra is a $\mathbb{C}$-algebra $(i: \mathbb{C} \hookrightarrow A,\ m(x \otimes y) = xy)$ with maps

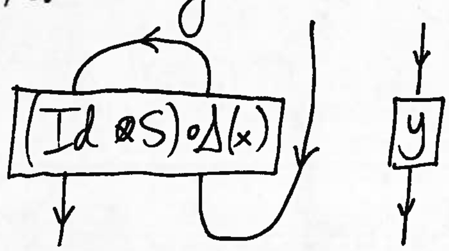
$\Delta: A \to A \otimes A$ (comultiplication)

$\varepsilon: A \to \mathbb{C}$ (counit)

$S: A \to A$ antihomomorphism $S(xy) = S(y)S(x)$

satisfying some nice relations.

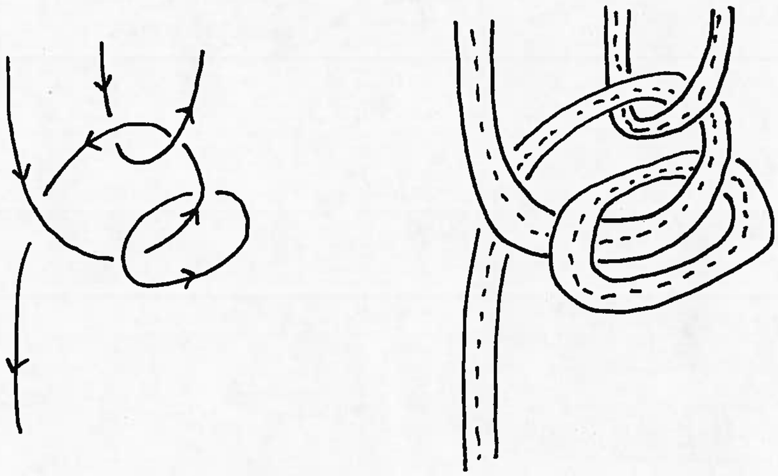
Diagrammatic Notation: e.g.



$= \left(m \circ P(Id \otimes S) \circ \Delta(x)\right) \otimes y \in A \otimes A$

$(P(x \otimes y) = y \otimes x)$

efn: A framed oriented tangle is a compact oriented 1-manifold T properly embedded in $\mathbb{R}\times\mathbb{R}\times[0,1]$ s.t. $\partial T \subset \{0\}\times\mathbb{R}\times\{0,1\}$, equipped with a framing: an inclusion of T as the midline of some union of embedded annuli and rectangles ("ribbons")



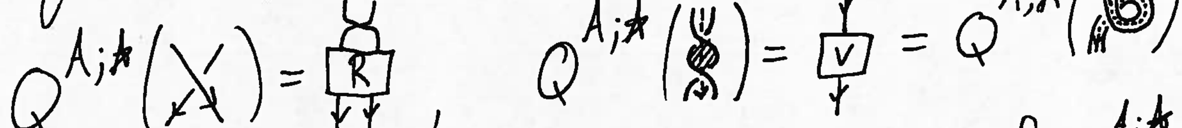
"black board framing"

Want: Interpret our diagrammatic notation literally; an assignment

$$Q^{A;\star} : \underset{\text{(isotopy classes of)}}{\text{Tangles}} \longrightarrow \bigoplus_n A^{\otimes n}.$$

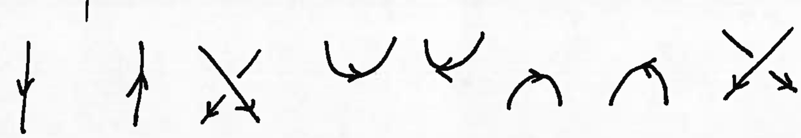
· These tangles have two pieces of information our diagrams don't: crossings and twists.

We thus require some specified $R \in A\otimes A$, $v\in A$ with



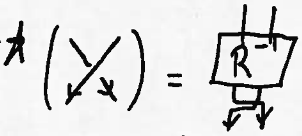
· We try to proceed as naturally as possible with our definition of $Q^{A;\star}$:

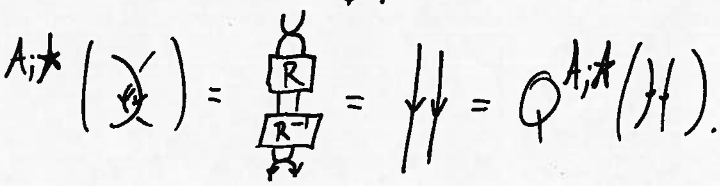
· Elementary Tangle Diagrams:

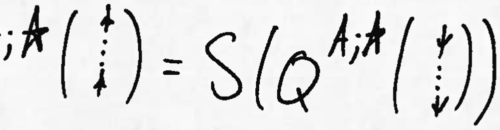
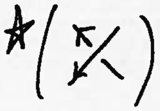
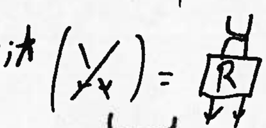
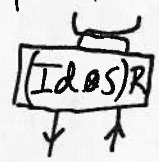


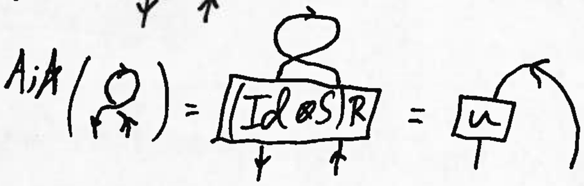
$$Q^{A;\star}(\downarrow) = \downarrow,\quad Q^{A;\star}(\uparrow) = \uparrow,\quad Q^{A;\star}(\smile) = \smile,\quad Q^{A;\star}(\frown) = \frown.$$

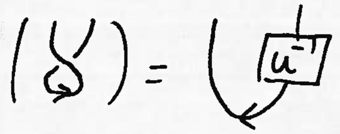
The rest are determined by our previous choices:

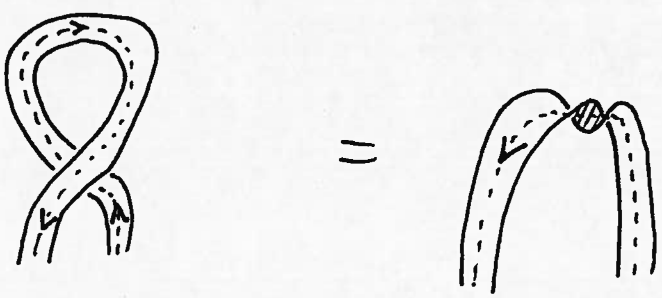
- We have an isotopy  (Reidemeister II), hence we must have $Q^{A;\star}$  so that

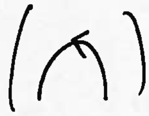
$Q^{A;\star}$ 

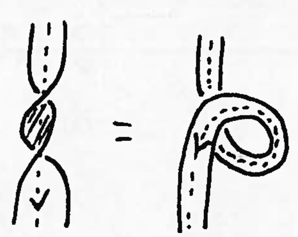
- <u>Guiding principle:</u> $Q^{A;\star}$ . In particular, $Q^{A;\star}$  $= (Id \otimes S)R$, because $Q^{A;\star}$ , and we apply S to the elt. on the reversed strand to get .

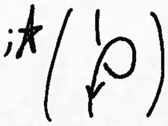
- Setting $u := mP(Id \otimes S)R$, we thus get $Q^{A;\star}$ .

- Similarly, $Q^{A;\star}$ . <u>Notice:</u> Under blackboard framing,

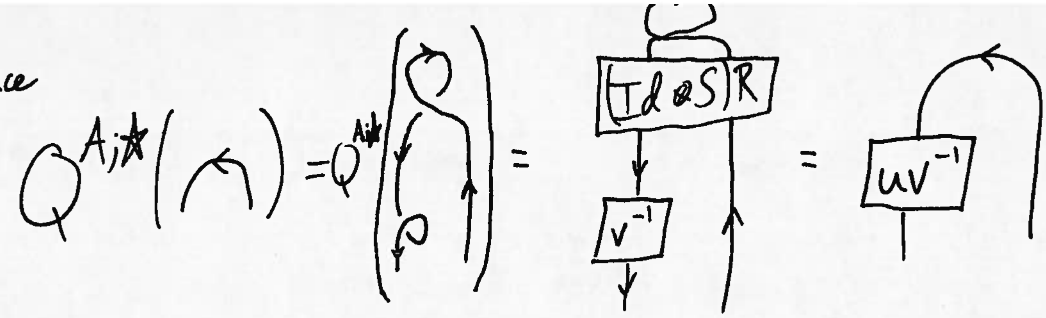


- To find $Q^{A;\star}$ , we must un-twist the above picture. The relevant un-twist is

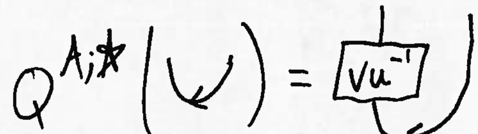


$Q^{A;\star}$  $= v^{-1}$.

hence



· Similarly,



· This finishes our definition of $Q^{A;\star}$ on elementary tangles, hence on all tangles!

<u>Defn</u>: A <u>ribbon Hopf algebra</u> is a triple (A, R, v), A a Hopf algebra, $R \in A \otimes A$ invertible, $v \in A$ invertible, satisfying ... (several relations that ensure $Q^{A;\star}$ is invariant under isotopies of tangles.)

<u>Operator Invariants</u>

· We want the arrows we've been drawing to be bona fide arrows in $\mathrm{Vect}_{\mathbb{C}}$!

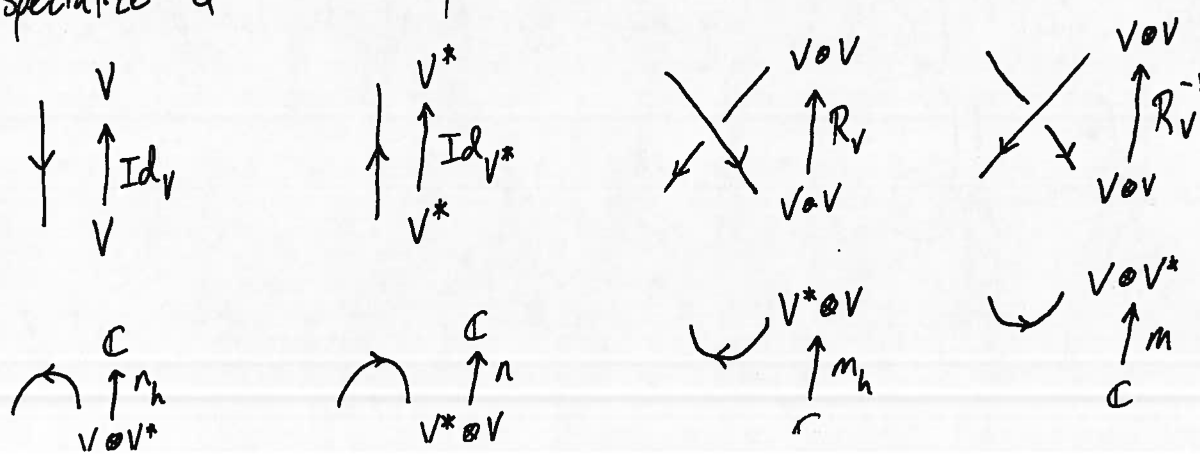
· Fix a representation $\rho : A \to \mathrm{End}_{\mathbb{C}}(V)$, and let

$$R_V := P \circ \big((\rho \otimes \rho)(R)\big) \in \mathrm{End}(V \otimes V), \quad h_V := \rho(uv^{-1}).$$

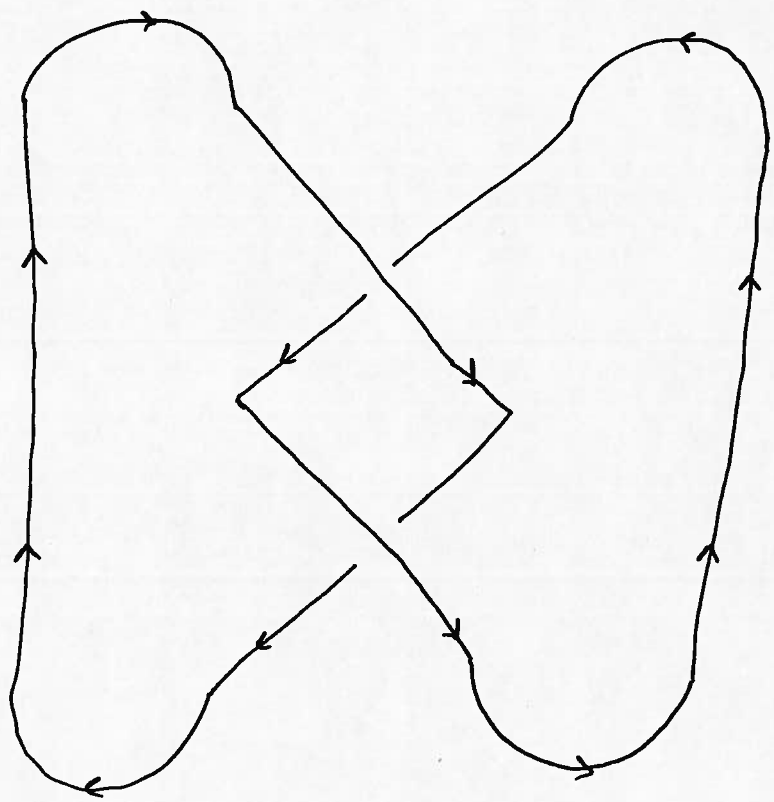
$$\big(P(x \otimes y) = y \otimes x\big)$$

· We can now specialize $Q^{A;\star}$ at the representation V to get:

<u>$Q^{A;V}$:</u>



· Example (Hopf Link):



$$\begin{tikzcd}
\mathbb{C} \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "n \otimes n_h"'] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "Id_{V^*} \otimes R_V \otimes Id_{V^*}"'] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "Id_{V^*} \otimes R_V \otimes Id_{V^*}"'] \\
\mathbb{C} \arrow[u, "m_h \otimes m"']
\end{tikzcd}$$

· Like $Q^{A;*}$, $Q^{A;V}(T)$ is invariant under isotopies of the tangle T.

Quantum sl_2

· $U_q(sl_2)$ is a ribbon Hopf algebra. The formulas for R and v are rather involved (See Ohtsuki's text).

$U_q(sl_2)$ has a 2-dimensional representation:

$$\rho_V(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho_V(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho_V(K) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Via explicit formulas for R, v, one computes:

$$R_V = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & q^{-1/4} & (q^{1/4}-q^{-3/4}) & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix}$$

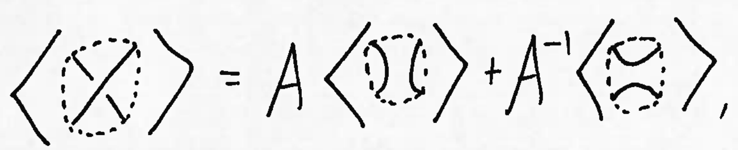
$$h_V = \rho_V(K) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Therefore, for links (i.e. tangles with no boundary), we have

$$Q^{U_q(sl_2);V}(L) \in \mathbb{C}[q^{\pm 1/4}].$$

. Up to sign, this is the Kauffman bracket!

Defn: We recursively define the Kauffman bracket $\langle L \rangle \in \mathbb{Z}[A^{\pm 1}]$ of a link diagram L:



$$\langle O \sqcup L' \rangle = (-A^2 - A^{-2})\langle L' \rangle.$$

Theorem: For a framed link L,

$$Q^{U_q(sl_2);V}(L) = (-1)^{\#L + f(L)} \langle L \rangle \Big|_{A=q^{1/4}},$$

where $\#L$ is the number of components of L, and $f(L)$ is the signed sum of the twists in the framing.

Defn: The *Jones polynomial* $J(L)$ of a link is

$$J(L) = (-A^{-3w(L)})\langle L\rangle,$$

where $w(L) = (\#\text{ of } \quad \text{crossings}) - (\#\text{ of } \quad \text{crossings}).$

Primary Reference: Ohtsuki, Tomotada; *Quantum Invariants*; 2002 World Scientific.