Notes on Kodaira-Spencer Theory

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1 Background

1.1 Complex Manifolds

Definition 1.1. A *complex manifold* of (complex) dimension n is a real $2n$-dimensional smooth manifold M with a maximal atlas of charts $\{\varphi_\lambda : M \supseteq U_\lambda \to U'_\lambda \subseteq \mathbb{C}^n\}_{\lambda\in\Lambda}$, where Λ is some index set, such that the coordinate transition functions are biholomorphisms.

Definition 1.2. A map $f : M \to N$ of complex manifolds of dimension m and n, respectively, is *holomorphic* if for every $p \in M$, there are charts $\varphi : U \to \mathbb{C}^m$ about p and $\psi : V \to \mathbb{C}^n$ with $V \supseteq f(U)$ such that $\psi\circ f\circ\varphi^{-1}$ is holomorphic.

Definition 1.3. A *holomorphic vector bundle* is a vector bundle whose local trivializations are biholomorphisms.

Definition 1.4. Let M be a complex n-manifold. We fix the following notation:

- $T_\mathbb{R}M$ is the tangent bundle to M as a real $2n$-manifold. It is a real $2n$-dimensional vector bundle with an $\mathbb{R}$-basis $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}$ at every point.
- $T_\mathbb{C}M = T_\mathbb{R}M \otimes_\mathbb{R} \mathbb{C}$ is the complexified tangent bundle to M. It is a complex $2n$-dimensional vector bundle with a $\mathbb{C}$-basis $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}$ at every point. It contains $T_\mathbb{R}M$ as the real $2n$-dimensional subbundle invariant pointwise under complex conjugation.
- TM is the holomorphic tangent bundle to M. It is a complex n-dimensional sub-vector bundle of $T_\mathbb{C}M$ with a $\mathbb{C}$-basis $\frac{\partial}{\partial z_1} := \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}),\dots,\frac{\partial}{\partial z_n} := \frac{1}{2}(\frac{\partial}{\partial x_n} - i\frac{\partial}{\partial y_n})$ at every point. The bundle TM is a holomorphic vector bundle.

1.2 Sheaves

In this section we follow the presentation of [4]. Let X be a topological space, and let $\mathbf{Top}(X)$ denote the category of open sets of X, whose arrows $U \to V$ are containments $U \supseteq V$.

Definition 1.5. Let $\mathbf{C}$ be a subcategory (not necessarily full) of $\mathbf{AbGrp}$ (for example, $\mathbf{Rings}$, $\mathbf{Vect}_k$, $\mathbb{C}-\mathbf{Algebras}$). A *sheaf* of objects of $\mathbf{C}$ on a topological space X is a functor $\mathscr{S} : \mathbf{Top}(X) \to \mathbf{C}$ satisfying properties (i) and (ii) below. Let us write $\mathrm{res}_{U,V} := \mathscr{S}(U \to V)$.

(i) (Gluing) If $U = \bigcup_{i\in I} U_i$, where I is some index set, then given $\sigma_i \in \mathscr{S}(U_i)$ for each i, if

$$\mathrm{res}_{U_i,U_i\cap U_j}\sigma_i = \mathrm{res}_{U_j,U_i\cap U_j}\sigma_i$$

for each i, j, then there is some $\sigma \in \mathscr{S}(U)$ such that $\mathrm{res}_{U,U_i}\sigma = \sigma_i$ for each i.

(ii) (Identity) If $U = \bigcup_{i\in I} U_i$, where I is some index set, and there are $\sigma, \sigma' \in \mathscr{S}(U)$ such that $\mathrm{res}_{U,U_i}\sigma = \mathrm{res}_{U,U_i}\sigma'$ for each i, then $\sigma = \sigma'$.

Elements σ of $\mathscr{S}(U)$ are called *sections* of $\mathscr{S}$ over U, and the maps $\mathrm{res}_{U,V}$ are called *restriction maps*, as motivated by the example below. A *homomorphism* $\mathscr{S} \to \mathscr{T}$ of sheaves is simply a natural transformation of functors $\mathscr{S} \Rightarrow \mathscr{T}$. We thus have a category $\mathbf{Sh_C}(X)$ of sheaves of objects of $\mathbf{C}$ on X, defined as a full subcategory of $\mathbf{C}^{\mathbf{Top}(X)}$.

Example 1.6 (Sheaf of sections)**.** Let $\pi : E \to M$ be a holomorphic vector bundle. Then $\mathscr{S}_E : \mathbf{Top}(M) \to \mathbf{Vect}_{\mathbb{C}}$ given by $\mathscr{S}_E(U) = \{\sigma : U \to E \mid \pi \circ \sigma = \mathrm{Id}_M,\ \sigma \text{ holomorphic}\}$ is a sheaf of $\mathbb{C}$-vector spaces. For a section $\sigma \in \mathscr{S}_E(U)$, the section $\mathrm{res}_{U,V}\sigma \in \mathscr{S}_E(V)$ is just the restriction $\sigma|_V$. In particular, $\Theta_M(U) := \mathscr{S}_{TM}(U)$ is the set of all holomorphic vector fields on U, and $\mathcal{O}_M(U) := \mathscr{S}_{M\times\mathbb{C}}(U)$ is the set of all holomorphic functions on U. In fact, $\mathcal{O}_M$ is a sheaf of $\mathbb{C}$-algebras.

Definition 1.7. Let X be a topological space, and let $\mathscr{R}$ be a sheaf of rings over X. A sheaf $\mathscr{S} : \mathbf{Top}(X) \to \mathbf{AbGrp}$ is called an *$\mathscr{R}$-module* if for every $U \in \mathbf{Top}(X)$, there is an action $\mathscr{R}(U) \times \mathscr{S}(U) \to \mathscr{S}(U)$ making $\mathscr{S}(U)$ a $\mathscr{R}(U)$-module, such that for every $U \supseteq V$, the diagram

$$\begin{array}{ccc} \mathscr{R}(U) \times \mathscr{S}(U) & \longrightarrow & \mathscr{S}(U) \\ {\scriptstyle \mathrm{res}_{U,V}\times\mathrm{res}_{U,V}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{U,V}} \\ \mathscr{R}(V) \times \mathscr{S}(V) & \longrightarrow & \mathscr{S}(V) \end{array}$$

commutes.

Note that Θ_M is an $\mathcal{O}_M$-module. Indeed, for every holomorphic vector bundle $E \to M$, the sheaf $\mathscr{S}_E$ is an $\mathcal{O}_M$-module. We will use this fact in §4.

Definition 1.8. Let X be a topological space and let $\mathscr{S}$ be any sheaf on X. Consider the full subcategory $\mathbf{Top}(X)_x$ of $\mathbf{Top}(X)$ whose objects are open neighborhoods of $x \in X$. Then $\mathbf{Top}(X)_x$ is a filtered category, i.e. a functor out of $\mathbf{Top}(X)_x$ is a directed system. Let $\mathscr{S}|_{\mathbf{Top}(X)_x}$ be the restriction of the functor $\mathscr{S}$ to the subcategory $\mathbf{Top}(X)_x$. Then we define the *stalk* of $\mathscr{S}$ at $x \in X$ to be the direct limit

$$\mathscr{S}_x := \varinjlim \mathscr{S}|_{\mathbf{Top}(X)_x}.$$

By the universal property of direct limits, a homomorphism $\mathscr{S} \to \mathscr{T}$ of sheaves induces a morphism $\mathscr{S}_x \to \mathscr{T}_x$ of stalks at every point $x \in X$.

Remark 1.9. Though much can be said of homological algebra with sheaves, suffice it to say that we call a sequence $0 \to \mathscr{R} \to \mathscr{S} \to \mathscr{T} \to 0$ of sheaves (where 0 denotes the sheaf of functions to $\{0\}$) *short exact* if the induced sequence $0_x \to \mathscr{R}_x \to \mathscr{S}_x \to \mathscr{T}_x \to 0_x$ is short exact for every $x \in X$, whenever our category is such that the latter statement is meaningful.

1.3 Čech Cohomology

In this section, we loosely follow the presentation of [2]. The Wikipedia page on Čech cohomology is also a good reference. Let X be a topological space, let R be a ring, and let $\mathscr{S}$ be a sheaf R-modules.

Definition 1.10. The *category* $\mathbf{Cov}(X)$ *of open covers of* X has as objects open covers of X, and has an arrow $\mathcal{U} \to \mathcal{V}$ whenever $\mathcal{V}$ is a refinement of $\mathcal{U}$. Note that since any two covers have a common refinement, $\mathbf{Cov}(X)$ is a filtered category, i.e., a functor out of $\mathbf{Cov}(X)$ is a directed system.

The idea of Čech cohomology is that an open cover is somewhat like a triangulation. More precisely, to every open cover $\mathcal{U}$ of X, there is an associated simplicial complex $\mathcal{N}(\mathcal{U})$ called the *nerve* of $\mathcal{U}$ whose 0-faces are members of $\mathcal{U}$, whose 1-faces are nonempty binary intersections of members of $\mathcal{U}$, whose 2-faces are nonempty trinary intersections of members of $\mathcal{U}$, and so on. Of course, given an arbitrary open cover $\mathcal{U}$ of X, the complex $\mathcal{N}(\mathcal{U})$ is not necessarily homotopy equivalent to X. One solution to this problem is to develop a notion of a *good* open cover, i.e. one such that we do have $\mathcal{N}(\mathcal{U}) \simeq X$. We discuss a different approach: we will develop a notion of cohomology groups $H^k(\mathcal{U}, \mathscr{S})$ for open covers $\mathcal{U}$ with coefficients in $\mathscr{S}$, and then we will take the direct limit of these groups along $\mathbf{Cov}(X)$. The idea is that as we refine a cover further and further, its nerve will become a better and better approximation of X, even if it is not necessarily ever homotopy equivalent to X.

Definition 1.11. Let $\mathcal{U} = \{U_i\}_{i \in I} \in \mathbf{Cov}(X)$, and let $k \geq 0$. The set $C^k(\mathcal{U}, \mathscr{S})$ of *k-cochains* has as elements collections $\{\sigma_{i_1 \cdots i_k}\}$ of sections $\sigma_{i_1 \cdots i_k} \in \mathscr{S}(U_{i_1 \cdots i_k})$ for each ordered k-tuple $(i_1, \ldots, i_k)$ of indices such that $U_{i_1 \cdots i_k} \coloneqq U_{i_1} \cap \cdots \cap U_{i_k} \neq \varnothing$, where $\{\sigma_{i_1 \cdots i_k}\}$ is required to be skew-symmetric in its indices:

$$\sigma_{i_{s(1)} \cdots i_{s(k)}} = \operatorname{sign}(s) \sigma_{i_1 \cdots i_k}, \qquad \forall s \in S_k.$$

Note that the formula

$$a\{\sigma_{i_1 \cdots i_k}\} + b\{\tau_{i_1 \cdots i_k}\} \coloneqq \{a\sigma_{i_1 \cdots i_k} + b\tau_{i_1 \cdots i_k}\}, \qquad a, b \in R$$

makes $C^k(\mathcal{U}, \mathscr{S})$ into an R-module.

We make $C^*(\mathcal{U}, \mathscr{S})$ into a cochain complex by defining the coboundary map

$$\begin{aligned} \delta : C^k(\mathcal{U}, \mathscr{S}) &\to C^{k+1}(\mathcal{U}, \mathscr{S}) \\ \{\sigma_{i_1 \cdots i_k}\} &\mapsto \{\tau_{j_1 \cdots j_{k+1}}\}, \end{aligned}$$

where

$$\tau_{j_1 \cdots j_{k+1}} \coloneqq - \sum_{\ell=1}^{k} (-1)^\ell \operatorname{res}_{U_{j_1 \cdots j_{\ell-1} j_{\ell+1} \cdots j_k}, U_{j_1 \cdots j_{k+1}}} (\sigma_{j_1 \cdots j_{\ell-1} j_{\ell+1} \cdots j_{k+1}}).$$

A simple computation verifies that δ is indeed a coboundary map, so that we have a cochain complex

$$C^0(\mathcal{U}, \mathscr{S}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{S}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathscr{S}) \xrightarrow{\delta} \cdots.$$

We define $H^k(\mathcal{U}, \mathscr{S})$ to be the kth cohomology group of this cochain complex.

We now must show that $H^k(-, \mathscr{S}) : \mathbf{Cov}(X) \to R - \mathbf{Mod}$ is a well-defined functor. Then, the direct limit

$$\check{H}^k(X, \mathscr{S}) \coloneqq \varinjlim H^k(-, \mathscr{S})$$

exists, because $R - \mathbf{Mod}$ is a cocomplete category. We will call this direct limit the *k-th Čech cohomology group of X with respect to $\mathscr{S}$.*

Remark 1.12. Note that this construction is unnecessary for $k = 0$, since by the gluing and identity axioms for sheaves, we have $H^0(\mathcal{U}, \mathscr{S}) = \mathscr{S}(X)$ for any open cover $\mathcal{U}$. Of course, this means that $\check{H}^0(X, \mathscr{S})$ is the limit of a constant functor, and so for every $\mathcal{U}$ we have $\check{H}^0(X, \mathscr{S}) = H^0(\mathcal{U}, \mathscr{S}) = \mathscr{S}(X)$.

We have already defined the functor $H^k(-, \mathscr{S})$ on the objects of $\mathbf{Cov}(X)$, and so we now define its action on the arrows of $\mathbf{Cov}(X)$. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a refinement of $\mathcal{U} = \{U_i\}_{i\in I}$. Let $f : J \to I$ be any function satisfying $V_j \subseteq U_{f(j)}$. We then define $\Pi^{\mathcal{U}}_{\mathcal{V},f} : C^k(\mathcal{U}) \to C^k(\mathcal{V})$ by

$$\Pi^{\mathcal{U}}_{\mathcal{V},f}(\{\sigma_{i_1\cdots i_k}\}) = \{\mathrm{res}_{U_{f(j_1)\cdots f(j_k)},V_{j_1\cdots j_k}}(\sigma_{f(j_1)\cdots f(j_k)})\}.$$

It is again a simple computation to show that $\Pi^{\mathcal{U}}_{\mathcal{V},f}$ descends to a map $P^{\mathcal{U}}_{\mathcal{V},f} : H^k(\mathcal{U}, \mathscr{S}) \to H^k(\mathcal{U}, \mathscr{S})$.

Lemma 1.13 (Lemma 3.2 of [2])**.** *The map $P^{\mathcal{U}}_{\mathcal{V},f}$ is independent of the choice of f.*

We therefore simply write $P^{\mathcal{U}}_{\mathcal{V}}$ for $P^{\mathcal{U}}_{\mathcal{V},f}$. Then we define

$$H^k(\mathcal{U} \to \mathcal{V}, \mathscr{S}) \coloneqq P^{\mathcal{U}}_{\mathcal{V}}.$$

Therefore we have a well-defined functor $H^k(-, \mathscr{S})$, and so the k-th Čech cohomology group of X with respect to $\mathscr{S}$ is well-defined. Finally, we will require the following theorem in §4.

Theorem 1.14 (Theorem 3.7 of [2])**.** *Given a short exact sequence $0 \to \mathscr{R} \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{T} \to 0$ of sheaves on a topological space X, we have a long exact sequence*

$$\check{H}^0(X, \mathscr{R}) \to \check{H}^0(X, \mathscr{S}) \to \check{H}^0(X, \mathscr{T}) \xrightarrow{\delta^*} \check{H}^1(X, \mathscr{R}) \to \cdots,$$

where the formula for the connecting homomorphism δ^* is as expected from the ordinary Snake Lemma for cohomology with constant coefficients:

$$\delta^*[\tau] = [\delta\{\sigma_i\}],$$

where $\tau \in \mathscr{T}(X)$, and $\{U_i\}_{i\in I}$ is a fine enough cover that we have a cochain $\{\sigma_i\} \in C^0(\{U_i\}_{i\in I}, \mathscr{S})$ satisfying $g(\sigma_i) = \mathrm{res}_{X,U_i}(\tau)$, and $[\,\cdot\,]$ denotes the Čech cohomology class of a cocycle.

2 Definition of the Kodaira-Spencer Mapping

The goal of Kodaira-Spencer theory is to develop a notion of a family of complex manifolds that vary with respect to some parameters $w_1, w_2, \ldots, w_m$, and then to say what it means to infinitesimally vary the complex manifolds in the direction of each $\frac{\partial}{\partial w_i}$. In this section we more or less follow the presentations found in [2] and [3].

2.1 Complex Analytic Families

Definition 2.1. A *complex analytic family of compact complex manifolds*, more concisely a *complex analytic family*, is a proper holomorphic map $\pi : M \to B$ from a complex $(m+n)$-manifold M to a complex m-manifold B such that:

(i) The derivative $(d\pi)_p : T_pM \to T_{\pi(p)}B$ is surjective for every $p \in M$.

(ii) The fiber $\pi^{-1}(b)$ is connected for every $b \in B$.

We call B the *base* or *parameter space* of the family, and any two fibers of π are called *deformations* of each other. Since it loses us no generality, we will typically assume that B is connected and π is surjective.

Example 2.2 (Family of Elliptic Curves)**.** Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The space $\mathbb{H}^2 \times \mathbb{C}$ has a $\mathbb{Z}^2$-action given by $(m, n).(\tau, z) = (\tau, z + m + n\tau)$. Obviously this action is by biholomorphisms and preserves fibers of the coordinate projection $\widetilde{\pi} : \mathbb{H}^2 \times \mathbb{C} \to \mathbb{H}^2$. Given any compact subset $K \subseteq \mathbb{H}^2 \times \mathbb{C}$, we have

$$\inf_{\substack{(m,n)\in\mathbb{Z}^2 \\ (\tau,z)\in K}} \mathrm{dist}((\tau, z), (m, n).(\tau, z)) = \min(\{1\} \cup \{|\tau| \mid (\tau, z) \in K\}) > 0,$$

and so this $\mathbb{Z}^2$-action is free and properly discontinuous. Therefore the quotient space $M = (\mathbb{H}^2 \times \mathbb{C})/\mathbb{Z}^2$ is a complex 2-manifold, and the projection $\widetilde{\pi}$ descends to a map $\pi : M \to \mathbb{H}^2$ with compact fibers. We claim that $\pi : M \to \mathbb{H}^2$ is a complex analytic family.

Since the $\mathbb{Z}^2$-action above is by biholomorphisms, π is a holomorphic map. Topologically, $\pi : M \to \mathbb{H}^2$ is a trivial torus bundle over $\mathbb{H}^2$, and hence π is proper and satisfies (i) and (ii) above. Note that the fiber $\pi^{-1}(\tau)$ is the elliptic curve determined by the lattice generated by 1 and τ. Indeed, this family of elliptic curves is "universal" in some sense: $\mathbb{H}^2$ is the Teichmüller space of the torus, and this family is the universal family of marked elliptic curves [1].

Remark 2.3. The fact that the family from Example 2.2 is a smooth fiber bundle is not a coincidence. Every complex analytic family is a smooth fiber bundle over its image ([2], Theorem 2.4). In particular, if the total space M of the family is connected, then all the fibers are diffeomorphic to each other. It is therefore only the complex manifold structure of the fibers that varies, not the diffeomorphism type.

2.2 Motivation for the Definition

Let $\pi : M \to B$ be a complex analytic family. Let $\varphi : U \to U' \subseteq \mathbb{C}^m$ be a coordinate chart for the base B, so that for some $b_0 \in B$ we have $\varphi(b_0) = (0, 0, \dots, 0)$. Since φ is a biholomorphism onto its image, it is reasonable to abuse notation by referring to a point $(w_1, w_2, \dots, w_m)$ of U, when we really mean a point $b \in U$ so that $\varphi(b) = (w_1, w_2, \dots, w_m)$. We will adopt this convention. Given an index set Λ, let us fix a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of $\pi^{-1}(U)$ by coordinate charts $\varphi_\lambda : U_\lambda \to U'_\lambda \subseteq \mathbb{C}^{m+n}$ so that the first m coordinates of $\varphi_\lambda(p)$ are equal to $\varphi(\pi(p))$. That is to say, there is some $\varphi'_\lambda : U_\lambda \to \mathbb{C}^n$ so that $\varphi_\lambda = (\varphi \circ \pi, \varphi'_\lambda) : U_i \to \mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$. As with points $(w_1, w_2, \dots, w_m) \in U$, we will abuse notation by referring to a point $(w_1, w_2, \dots, w_m, z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda)$ of U_λ, when we really mean a point $x \in U_\lambda$ so that $\varphi_\lambda(x) = (w_1, w_2, \dots, w_m, z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda)$.

Given U_λ, U_μ such that $U_\lambda \cap U_\mu \neq \varnothing$, consider the coordinate transition function

$$f^{\lambda\mu} = (g_1^{\lambda\mu}, g_2^{\lambda\mu}, \dots, g_m^{\lambda\mu}, f_1^{\lambda\mu}, f_2^{\lambda\mu}, \dots, f_n^{\lambda\mu}) : \varphi_\lambda(U_\lambda \cap U_\mu) \to \varphi_\mu(U_\lambda \cap U_\mu).$$

Since $f^{\lambda\mu}$ is a coordinate transition function, we have $\varphi_\mu = f^{\lambda\mu} \circ \varphi_\lambda$. In particular, for $1 \le i \le n$, we have (recalling our abuse of notation):

$$z_i^\mu = f_i^{\lambda\mu}(w_1, w_2, \dots, w_m, z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda). \tag{1}$$

Notice that φ_λ and φ_μ do not differ in the first m coordinates: we have $\varphi_\lambda = (\varphi\circ\pi, \varphi'_\lambda)$ and $\varphi_\mu = (\varphi\circ\pi, \varphi'_\mu)$. Therefore $g_i^{\lambda\mu} = \mathrm{Id}_\mathbb{C}$ for each $1 \le i \le m$. Therefore, only the functions $f_i^{\lambda\mu}$ are interesting.

Let $F_0 = \pi^{-1}(0, 0, \dots, 0)$. As we vary, say, the first coordinate of $(0, 0, \dots, 0)$, how does the complex manifold structure of the fiber differ from the complex manifold structure of F_0? For small enough $w_1 \in \mathbb{C}$, notice that we have $F_0 \cap U_\lambda \neq \varnothing$ if and only if $\pi^{-1}(w_1, 0, \dots, 0) \cap U_\lambda \neq \varnothing$, and $F_0 \cap U_\lambda \cap U_\mu \neq \varnothing$ if and only if $\pi^{-1}(w_1, 0, \dots, 0) \cap U_\lambda \cap U_\mu \neq \varnothing$. Let $\Lambda' \subseteq \Lambda$ be the set of λ such that $F_0 \cap U_\lambda \neq \varnothing$. Then the complex structures on F_0 and $\pi^{-1}(w_1, 0, \dots, 0)$ are determined by the atlases of charts $\{\varphi_\lambda|_{F_0\cap U_\lambda}\}_{\lambda\in\Lambda'}$ and $\{\varphi_\lambda|_{\pi^{-1}(w_0,0,\dots,0)\cap U_\lambda}\}_{\lambda\in\Lambda'}$, respectively. If we had chosen a fine enough open cover, we would even have that $F_0 \cap U_\lambda$ and $\pi^{-1}(w_1, 0, \dots, 0) \cap U_\lambda$ are biholomorphic for each $\lambda \in \Lambda'$ (cf. §4.1(b) of [2]). Now we are in the position to say: the difference between the complex structure on F_0 and that on $\pi^{-1}(w_1, 0, \dots, 0)$ is entirely encoded in the data of the transition functions $f^{\lambda\mu}$ for $\lambda, \mu \in \Lambda'$. It is therefore reasonable to say that the derivatives

$$\left\{ \frac{\partial f_i^{\lambda\mu}}{\partial w_1}(0, 0, \dots, 0, z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda) \right\}_{\lambda,\mu\in\Lambda'}^{i=1,\dots,n}$$

give us first-order information about how the complex structures of the fibers of π vary as we walk from F_0 towards $\pi^{-1}(w_1, 0, \dots, 0)$.

2.3 The Kodaira Spencer Mapping

We continue to use the notation laid out in the previous subsection.

Definition 2.4. The *infinitesimal deformation* of F_0 in the w_k direction is the 1-cochain of holomorphic vector fields

$$\left\{\theta^{(k)}_{\lambda\mu}\right\} \in C^1(\{F_0 \cap U_\lambda\}_{\lambda\in\Lambda'}, \Theta_{F_0})$$
$$\theta^{(k)}_{\lambda\mu} = \sum_{i=1}^{n} \frac{\partial f_i^{\lambda\mu}}{\partial w_k}(0,0,\ldots,0,z_1^\lambda,z_2^\lambda,\ldots,z_n^\lambda)\frac{\partial}{\partial z_i^\mu} \in \Theta_{F_0}(F_0\cap U_\lambda\cap U_\mu).$$

Lemma 2.5. *The cochain* $\left\{\theta^{(k)}_{\lambda\mu}\right\}$ *is a cocycle.*

Proof. Consider $U_\alpha, U_\beta, U_\gamma$ such that $F_0 \cap U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. We will consider the restriction of $\theta^{(k)}_{\alpha\gamma}$ to this intersection, and we will everywhere elide the notation "res" for the restriction map of the sheaf Θ_{F_0}. For $1 \le i \le n$, Equation (1) gives us that

$$\begin{aligned}
f_i^{\alpha\gamma}(0,\ldots,0,z_1^\alpha,\ldots,z_n^\alpha) &= z_i^\gamma \\
&= f_i^{\beta\gamma}(0,\ldots,0,z_1^\beta,\ldots,z_n^\beta) \\
&= f_i^{\beta\gamma}(0,0,\ldots,0,f_1^{\alpha\beta}(0,\ldots,0,z_1^\alpha,\ldots,z_n^\alpha),\ldots,f_n^{\alpha\beta}(0,\ldots,0,z_1^\alpha,\ldots,z_n^\alpha)).
\end{aligned}$$

As shorthand, let us write $z^\cdot = (z_1^\cdot, z_2^\cdot, \ldots, z_n^\cdot)$, and let $\overline{0}$ be the zero vector in $\mathbb{C}^m$. Differentiating the first and last parts of the above equation gives

$$\frac{\partial f_i^{\alpha\gamma}}{\partial w_k}(\overline{0},z^\alpha) = \frac{\partial f_i^{\beta\gamma}}{\partial w_k}(\overline{0},z^\beta) + \sum_{j=1}^{n}\frac{\partial f^{\beta\gamma}}{\partial z_j^\beta}(\overline{0},z^\beta)\cdot\frac{\partial f^{\alpha\beta}}{\partial w_k}(\overline{0},z^\alpha).$$

This gives the following identity of holomorphic vector fields on $F_0 \cap U_\alpha \cap U_\beta \cap U_\gamma$.

$$\begin{aligned}
\theta^{(k)}_{\alpha\gamma} = \sum_{i=1}^{n}\frac{\partial f_i^{\alpha\gamma}}{\partial w_k}(\overline{0},z^\alpha)\frac{\partial}{\partial z_i^\gamma} &= \sum_{i=1}^{n}\frac{\partial f_i^{\beta\gamma}}{\partial w_k}(\overline{0},z^\beta)\frac{\partial}{\partial z_i^\gamma} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f^{\beta\gamma}}{\partial z_j^\beta}(\overline{0},z^\beta)\cdot\frac{\partial f^{\alpha\beta}}{\partial w_k}(\overline{0},z^\alpha)\frac{\partial}{\partial z_i^\gamma} \\
&= \sum_{i=1}^{n}\frac{\partial f_i^{\beta\gamma}}{\partial w_k}(\overline{0},z^\beta)\frac{\partial}{\partial z_i^\gamma} + \sum_{i=1}^{n}\frac{\partial f^{\alpha\beta}}{\partial w_k}(\overline{0},z^\alpha)\frac{\partial}{\partial z_i^\beta} \\
&= \theta^{(k)}_{\beta\gamma} + \theta^{(k)}_{\alpha\beta}.
\end{aligned}$$

Thus,

$$0 = \left\{\theta^{(k)}_{\beta\gamma} - \theta^{(k)}_{\alpha\gamma} + \theta^{(k)}_{\alpha\beta}\right\} = \delta\left\{\theta^{(k)}_{\lambda\mu}\right\},$$

and so $\left\{\theta^{(k)}_{\lambda\mu}\right\}$ is a cocycle, as desired. ☐

Let us denote by $\left[\theta^{(k)}_{\lambda\mu}\right]$ the Čech cohomology class determined by $\left\{\theta^{(k)}_{\lambda\mu}\right\}$. A direct computation shows that this cohomology class is independent of the choice of open covering $\{U_\lambda\}_{\lambda\in\Lambda'}$ used to define $\theta^{(k)}_{\lambda\mu}$ (see §4.2(a) of [2]). We now define the Kodaira-Spencer mapping.

Definition 2.6. The linear map

$$\rho_0 : T_0B \to \check{H}^1(F_0, \Theta_{F_0})$$
$$\sum_{k=1}^{m} a_k \frac{\partial}{\partial w_k} \mapsto \sum_{k=1}^{m} a_k \left[\theta^{(k)}_{\lambda\mu}\right]$$

is called the *Kodaira-Spencer mapping.*

Of course, the point $(0, 0, \ldots, 0)$ is not at all special in what we have done, it is merely convenient. We therefore have for every $b = (w_1, w_2, \ldots, w_m) \in U$ and $F_b = \pi^{-1}(b)$ a Kodaira-Spencer mapping

$$\begin{aligned} \rho_b : T_b B &\to \check{H}^1(F_b, \Theta_{F_b}) \\ \sum_{k=1}^m a_k \frac{\partial}{\partial w_k} &\mapsto \sum_{k=1}^m a_k \left[\theta_{\lambda\mu}^{(k)}(b)\right], \end{aligned}$$

where

$$\theta_{\lambda\mu}^{(k)}(b) = \sum_{i=1}^n \frac{\partial f_i^{\lambda\mu}}{\partial w_k}(w_1, w_2, \ldots, w_m, z_1^\lambda, z_2^\lambda, \ldots, z_n^\lambda) \frac{\partial}{\partial z_i^\mu} \in \Theta_{F_b}(F_b \cap U_\lambda \cap U_\mu).$$

3 Local Triviality of Families

In this section, we mostly follow [2].

If $\left[\theta_{\lambda\mu}^{(k)}(b)\right]$ really does represent first-order information about how the complex structures of the fibers of π vary, then we might expect that $\left[\theta_{\lambda\mu}^{(k)}(b)\right] = 0$ for all $b \in B$ and $1 \leq k \leq m$ if and only if the family $\pi : M \to B$ is in some sense "locally trivial." In fact, this is not true in general, but is only true given an additional assumption. We first formulate what it means to be locally trivial.

Definition 3.1. A complex analytic family $\pi : M \to B$ is called *locally trivial* if for every $b_0 \in B$, there is an open neighborhood U of b_0 and a biholomorphism $\varphi : \pi^{-1}(U) \to \pi^{-1}(b_0) \times U$ taking $\pi^{-1}(b)$ biholomorphically onto $\pi^{-1}(b_0) \times \{b\}$ for every $b \in U$.

It is obvious that for a family of the form $F \times B \to B$, where F is a complex n-manifold, we have $\left[\theta_{\lambda\mu}^{(k)}(b)\right] = 0$ for all $b \in B$ and $1 \leq k \leq m$. Therefore $\left[\theta_{\lambda\mu}^{(k)}(b)\right] = 0$ for all $b \in B$ and $1 \leq k \leq m$ whenever $\pi : M \to B$ is a locally trivial family. To get the converse, we need an assumption on the dimensions of the cohomology groups of the fibers of our family.

Theorem 3.2 (Theorem 4.6 of [2])**.** *Let $\pi : M \to B$ be a complex analytic family with fibers $F_b = \pi^{-1}(b)$. If $\dim \check{H}^1(F_b, \Theta_{F_b})$ is independent of b, and if $\left[\theta_{\lambda\mu}^{(k)}(b)\right] = 0$ for all $b \in B$ and $1 \leq k \leq m$, then $\pi : M \to B$ is locally trivial.*

We do have some *a priori* control on $\dim \check{H}^1(F_b, \Theta_{F_b})$ though, thanks to the following theorem.

Theorem 3.3 (Theorem 4.4 of [2])**.** *Let $\pi : M \to B$ be a complex analytic family with fibers $F_b = \pi^{-1}(b)$. Then $\dim \check{H}^1(F_b, \Theta_{F_b})$ is an upper semicontinuous function of b.*

4 The Kodaira-Spencer Mapping as a Connecting Homomorphism

Though we have developed the Kodaira-Spencer mapping from the ground up in order to emphasize that it is what one would intuitively expect an infinitesimal deformation of complex structure to look like, the mapping also simply pops out of a completely natural short exact sequence of sheaves that arises from the projection map $\pi : M \to B$. Let $\pi : M \to B$ be a complex analytic family, let $b \in B$, and let $F_b = \pi^{-1}(b)$. We have a fiberwise linear map (or, if you like, a vector bundle morphism covering the map $F_b \to \{b\}$)

$$\begin{aligned} d\pi : TM|_{F_b} &\to T_b B \\ v &\mapsto d\pi(v). \end{aligned}$$

Lemma 4.1. *The map $d\pi : TM_{F_b} \to T_b B$ induces a short exact sequence of $\mathcal{O}_{F_b}$-modules*

$$0 \to \Theta_{F_b} \to \mathscr{S}_{TM|_{F_b}} \xrightarrow{d\pi} T_b B \otimes \mathcal{O}_{F_b} \to 0,$$

where $T_b B \otimes \mathcal{O}_{F_b}$ is the sheaf such that $(T_b B \otimes \mathcal{O}_{F_b})(U) = T_b B \otimes_{\mathbb{C}} \mathcal{O}_{F_b}(U)$ for every $U \in \mathbf{Top}(F_b)$.

Proof. Exactness at Θ_{F_b}: We have $T_pF_b \subseteq T_pM$ for every $p \in F_b$, so a germ of holomorphic vector fields at p is at the same time a germ of sections of $TM|_{F_b}$.

Exactness at $T_bB \otimes \mathcal{O}_{F_b}$: We know by Remark 2.3 that $\pi : M \to B$ is a fiber bundle, and so there is a small enough neighborhood $U \subseteq M$ about any $p \in F_b$, biholomorphic to some $U' \times U'' \subseteq \mathbb{C}^m \times \mathbb{C}^n$, so that $\pi|_U$ is just projection onto the first m coordinates. Therefore we may express any germ of sections of $TM|_{F_b}$ at $p \in F_b$ as $\sum_{k=1}^m [a_k] \frac{\partial}{\partial w_k} + \sum_{i=1}^n [b_k] \frac{\partial}{\partial z_i}$, where $[a_k], [b_i] \in \mathcal{O}_{F_p}$ for each k, i. We have

$$d\pi \left(\sum_{k=1}^m [a_k] \frac{\partial}{\partial w_k} + \sum_{i=1}^n [b_k] \frac{\partial}{\partial z_i} \right) = \sum_{k=1}^m \frac{\partial}{\partial w_k} \otimes [a_k],$$

and hence $(\mathscr{S}_{TM|_{F_b}})_p \xrightarrow{d\pi} (T_bB \otimes \mathcal{O}_{F_b})_p$ is surjective for every $p \in F_b$.

Exactness at $\mathscr{S}_{TM|_{F_b}}$: The above formula for $d\pi$ shows that $(\Theta_{F_b})_p$ is precisely the kernel of $d\pi$ at every $p \in F_b$. □

Remark 4.2. Since F_b is compact, it follows from the Maximum Modulus Principle that $\mathcal{O}_{F_b}(F_b) \cong \mathbb{C}$. Therefore, by Remark 1.12, $\check{H}^0(F_b, T_bB \otimes \mathcal{O}_{F_b}) \cong T_bB$.

By Theorem 1.14, we have a long exact sequence

$$\check{H}^0(F_b, \Theta_{F_b}) \to \check{H}^0(F_b, \mathscr{S}_{TM|_{F_b}}) \to T_bB \xrightarrow{\delta^*} \check{H}^1(F_b, \Theta_{F_b}) \to \cdots,$$

where for each $1 \leq k \leq m$,

$$\frac{\partial}{\partial w_k} \overset{\delta^*}{\longmapsto} [\delta\{\sigma_\lambda\}], \quad \sigma_\lambda \in \mathscr{S}_{TM|_{F_b}}(F_b \cap U_\lambda), \, d\pi(\sigma_\lambda) = \frac{\partial}{\partial w_k},$$

where $\{U_\lambda\}_{\lambda \in \Lambda'}$ is some satisfactory open covering of M as in §2.2. Note that σ_λ is not guaranteed to be unique, but certainly $\left(\frac{\partial}{\partial w_k}\right)_\lambda := d\left(\varphi_\lambda^{-1}\right)\left(\frac{\partial}{\partial w_k}\right)$ works as a choice for σ_λ. Then we have $\delta\{\sigma_\lambda\} = \{\tau_{\lambda\mu}\}$, where

$$\begin{aligned}
\tau_{\lambda\mu} &= \sigma_\lambda - \sigma_\mu \\
&= \left(\frac{\partial}{\partial w_k}\right)_\lambda - \left(\frac{\partial}{\partial w_k}\right)_\mu \\
&= \left(\frac{\partial w_k}{\partial w_k}\left(\frac{\partial}{\partial w_k}\right)_\mu + \sum_{i=1}^n \frac{\partial f_i^{\lambda\mu}}{\partial w_k}\frac{\partial}{\partial z_i^\mu}\right) - \left(\frac{\partial}{\partial w_k}\right)_\mu \\
&= \sum_{i=1}^n \frac{\partial f_i^{\lambda\mu}}{\partial w_k}\frac{\partial}{\partial z_i^\mu} \\
&= \theta_{\lambda\mu}^{(k)}(b).
\end{aligned}$$

We have therefore proved the following theorem (cf. e.g. Proposition 1 of [5] and its proof).

Theorem 4.3. *Given a complex analytic family $\pi : M \to B$, for each fiber $\pi^{-1}(b) = F_b$, the Kodaira-Spencer mapping ρ_b is the connecting homomorphism in the long exact sequence*

$$\check{H}^0(F_b, \Theta_{F_b}) \to \check{H}^0(F_b, \mathscr{S}_{TM|_{F_b}}) \to \check{H}^0(F_b, T_bB \otimes \mathcal{O}_{F_b}) \cong T_bB \xrightarrow{\delta^* = \rho_b} \check{H}^1(F_b, \Theta_{F_b}) \to \cdots.$$

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