Introduction to Khovanov Homology

Overview:

$$\begin{tikzcd}
\text{Category of } (n-1)\text{-manifolds } \& \ n\text{-cobordisms} \arrow[r, "\text{TQFT}"] & \mathrm{Vect}_{\mathbb{k}}
\end{tikzcd}$$

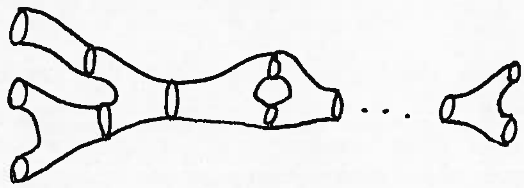
$$\Bigg\}\ \text{Embed this idea into Euclidean space and specialize to } n=2.$$

$$\begin{tikzcd}
\text{Category of } \left(\begin{array}{c}\text{Links and}\\ \text{Knots in } \mathbb{R}^3\end{array}\right) \ \& \ (\text{embedded surfaces in } \mathbb{R}^4) \arrow[rr, "\text{Khovanov Homology}"] \arrow[dr, "\text{Inspiration from Jones polynomial}"'] & \arrow[d, phantom, "\circlearrowright"] & (\text{bi-graded})\ \mathrm{Vect}_{\mathbb{k}} \\
& \text{Category of formal chain complexes of } 1\text{-manifolds } \& \ 2\text{-cobordisms} \arrow[ur, "\text{TQFT}"'] &
\end{tikzcd}$$

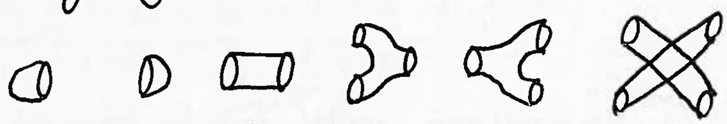
Topological Quantum Field Theories (TQFT)

- We consider $(n-1)$-manifolds Σ_1, Σ_2 to be "space" and an n-manifold M with $\partial M = \Sigma_1 \amalg \Sigma_2$ to be "spacetime," the evolution from the initial space-configuration Σ_1 to the final one Σ_2.
- A quantum field theory (QFT) assigns to each "space" a vectorspace of "fields," and to each "spacetime" a "time-evolution operator," a linear map between these vectorspaces.
- <u>Definition</u>: The category $n\mathrm{Cob}$ has as objects $(n-1)$-dimensional closed oriented manifolds, and as morphisms n-dimensional oriented cobordisms between them up to boundary-fixing diffeomorphism.

· In 2Cob, every arrow can be decomposed e.g.

 (read left to right)

as a gluing of the six elementary cobordisms



(Note: swapping  must be included because it is not equivalent to  ; our diffeomorphisms fix boundary!)

· <u>Definition:</u> A functor $n\mathrm{Cob} \xrightarrow{F} \mathrm{Vect}_{\mathbb{k}}$ satisfying $F(\Sigma_1 \sqcup \Sigma_2) = F(\Sigma_1) \otimes F(\Sigma_2)$ is called an <u>n-TQFT</u>.

· A 2-TQFT sends these elementary cobordisms to the structural maps for a ~~[illegible]~~ simultaneous algebra and coalgebra*: Let $A = F(S^1)$.

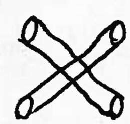
*<u>Note:</u> <u>not</u> a bialgebra!

unit  $\eta: \mathbb{k} \to A$

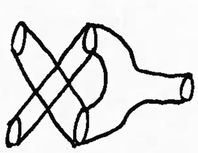
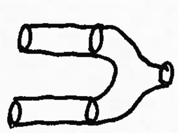
counit  $\varepsilon: A \to \mathbb{k}$

multiplication  $m: A \otimes A \to A$

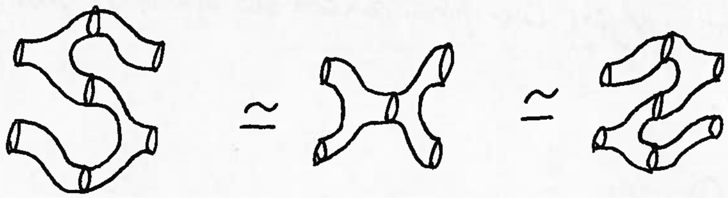
comultiplication  $\Delta: A \to A \otimes A$

 $x \otimes y \mapsto y \otimes x$

· We have various relations induced by equivalence of cobordisms, e.g.

 $\simeq$  $\Rightarrow$ multiplication is commutative in A

· In particular, the Frobenius relation holds:



$$(\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id}) \quad = \quad \Delta \circ m \quad = (m \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta)$$

· Definition: A simultaneous algebra and coalgebra satisfying the Frobenius relation is called a Frobenius algebra.

· Theorem: Every commutative & cocommutative Frobenius algebra determines a 2-TQFT. (There is a canonical equivalence of categories $2TQFT_{\mathbb{k}} \simeq cFrob_{\mathbb{k}}$)

Khovanov Homology

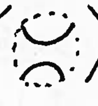
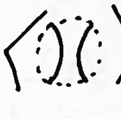
· Question: What if instead of 2Cob I consider the category $Cob^4_{/i}(\emptyset)$:

Objects: Embedded closed 1-mfds in $\mathbb{R}^3$ (i.e. links)

Morphisms: Embedded 2-cobordisms in $\mathbb{R}^4$ up to isotopy.

?

· Inspiration: The Jones polynomial $\hat{J}(L) \in \mathbb{Z}[q^{\pm 1}]$ of a link L is given by modifying the Kauffman bracket polynomial, defined recursively:

$\Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle - q \Big\langle$  $\Big\rangle$, $\langle \bigcirc \sqcup L' \rangle = (q+q^{-1}) \langle L' \rangle$, $\langle (\text{empty}) \rangle = 1$.

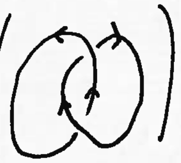
We orient our links, then call  a positive crossing,  a negative crossing, and set:

n_+ = number of positive crossings

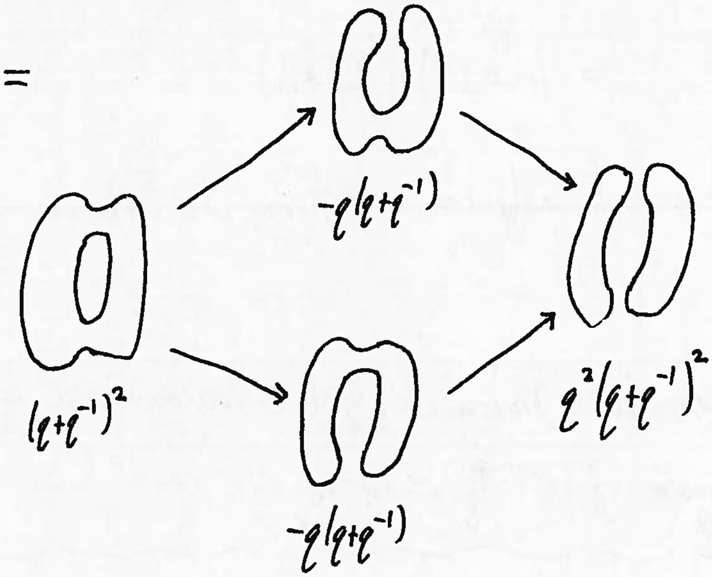
n_- = number of negative crossings.

Then we define: $\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$.

- We can evaluate $\hat{J}(L)$ by first applying the first relation as much as possible to obtain a <u>cube of resolutions</u>, e.g.

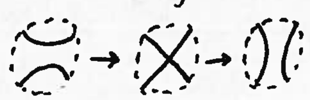
$\hat{J}\Big($  $\Big) =$

$n_+ = 0$
$n_- = 2$



and then adding everything up: $(q+q^{-1})^2 - 2q(q+q^{-1}) + q^2(q+q^{-1})^2$

and then multiplying by $(-1)^{n_-} q^{n_+ - 2n_-}$: $(-1)^2 q^{-4}\left((q+q^{-1})^2 - 2q(q+q^{-1}) + q^2(q+q^{-1})^2\right) = q^{-6} + q^{-4} + q^{-2} + 1$.

- Let's view the cube of resolutions as a bona fide commutative diagram in 2Cob, where every arrow is a saddle cobordism  (or vice versa).
- Applying a TQFT to this diagram will turn it into a commutative diagram in Vect_k!
- Let V be the 2-dimensional graded $\mathbb{C}$-vector space spanned in degree 1 by a vector $\mathbb{1}$ and spanned in degree -1 by a vector x. Define a Frobenius algebra structure on V by setting:

<u>Multiplication</u>	<u>Comultiplication</u>	<u>Unit</u>	<u>Counit</u>
$x \cdot \mathbb{1} = \mathbb{1} \cdot x = x$,	$\Delta(\mathbb{1}) = \mathbb{1} \otimes x + x \otimes \mathbb{1}$	$\eta(1) = \mathbb{1}$	$\varepsilon(x) = 1$,
$\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$, $x \cdot x = 0$	$\Delta(x) = x \otimes x$		$\varepsilon(\mathbb{1}) = 0$.

- This determines a TQFT $\mathcal{F}_V$, and we apply this to our diagram.
- <u>Note:</u> We call the grading of V the <u>q-grading</u>. The q-dimension of $V = \mathcal{F}_V(S^1)$ is $q + q^{-1}$, the same as $\hat{J}(\bigcirc)$!

· We now have the commutative diagram, continuing our example:

$$\begin{tikzcd}
& V \arrow[dr, "\Delta"] & \\
V^{\otimes 2} \arrow[ur, "m"] \arrow[dr, "m"'] & & V^{\otimes 2} \\
& V \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

* in general, we negate one arrow per commutative face of the cube.

· If we take the direct sum along columns, this will not give us a chain complex, but since the diagram commutes, negating one of the arrows* will turn the result into a chain complex:

$$\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to V^{\otimes 2} \xrightarrow{m \oplus m} V \oplus V \xrightarrow{\Delta \oplus -\Delta} V^{\otimes 2} \to 0 \to 0 \to \cdots$$

· Before we take homology, let's apply a modification analogous to that done for the Jones polynomial:
- move down in homological degree by n_-
- move up in q-grading by $n_+ - 2n_-$

$$\cdots \to 0 \to 0 \to 0 \to V^{\otimes 2}\{-4\} \xrightarrow{m \oplus m} \underset{\{-4\}}{V \oplus V} \xrightarrow{\Delta \oplus -\Delta} \underset{\{-4\}}{V^{\otimes 2}} \to 0 \to 0 \to 0 \to 0 \to \cdots$$

where $\{-4\}$ denotes the q-grading shift.

· Theorem: (Khovanov) The homology of this chain complex is an invariant of the link L.

· We call this the Khovanov homology of L, denoted $H^*_{Kh}(L)$.

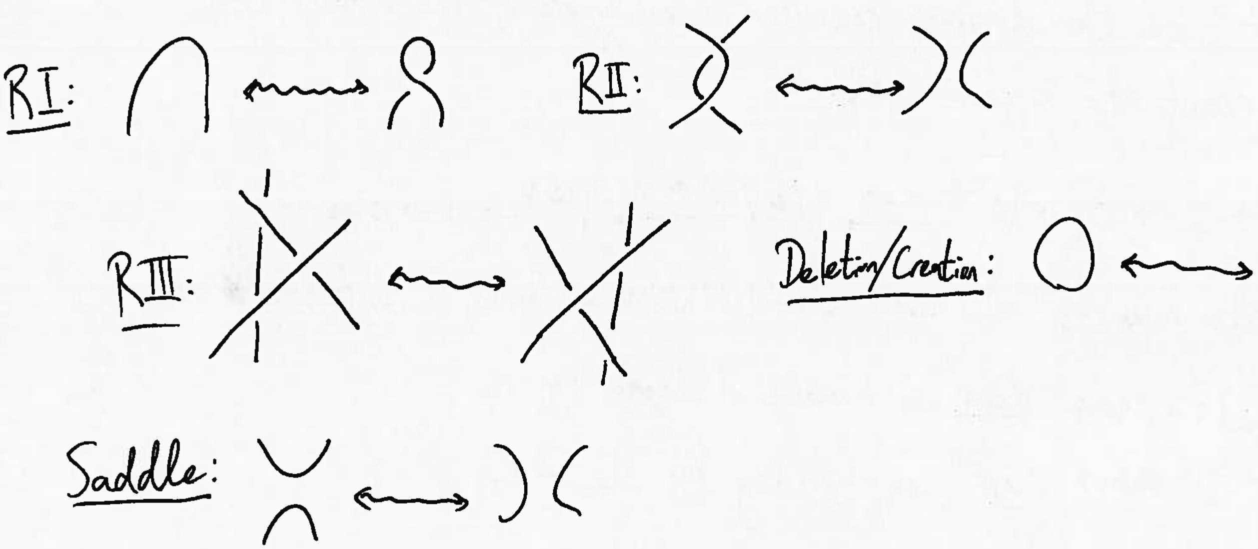
· The graded Euler characteristic $\chi_q(H^*_{Kh}(L)) = \sum_{i \in \mathbb{Z}} (-1)^i \, q\text{-dim}(H^i_{Kh}(L))$ is equal to $\hat{J}(L)$.

Functoriality

· Since each $H^i_{Kh}(L)$ is q-graded, and $H^*_{Kh}(L)$ is graded by homological degree, we have a map:

$$\mathrm{obj}(\mathrm{Cob}_{/i}(\emptyset)) \xrightarrow{H^*_{Kh}} \mathrm{obj}((\text{bi-graded})\ \mathrm{Vect}_{\mathbb{C}})$$

- Similarly to our factorization of morphisms in 2Cob into compositions of elementary cobordisms, the morphisms in $Cob^4_{/i}(\emptyset)$ can all be built up by local "movies" (i.e. we think of the 4^{th} dimension in $\mathbb{R}^4$ as "time"):



- With some effort, one can write down the morphisms of (bi-graded) $Vect_{\mathbb{C}}$ determined by these local movies.

- One must ensure that these definitions respect isotopy of 2-cobordisms in $\mathbb{R}^4$, i.e. one must show that H^*_{Kh} acts trivially on the kernel of the projection $Cob^4(\emptyset) \longrightarrow Cob^4_{/i}(\emptyset)$. This kernel is generated by the so-called "movie moves" of Carter and Saito.

Applications

- Theorem (Kronheimer-Mrowka, 2011): If $H^*_{Kh}(L) = H^*_{Kh}(\text{unknot})$, then L is the unknot.

Rasmussen's s-invariant:

- We consider an alternative knot homology theory determined by a TQFT defined by Eun Soo Lee:

$$\mathbb{1}^2 = x^2 = \mathbb{1}, \quad \mathbb{1}\cdot x = x \cdot \mathbb{1} = x,$$

$$\Delta'(\mathbb{1}) = \mathbb{1}\otimes x + x \otimes \mathbb{1}, \quad \Delta'(x) = x \otimes x + \mathbb{1}\otimes\mathbb{1}.$$

- Theorem (Rasmussen, 2004): There is a spectral sequence with E^1 term $H^*_{Kh}(L)$ that converges to $H^*_{Lee}(L)$. The E^1 and higher terms are invariants of L.

- The E^∞ term is isomorphic to $\mathbb{Q}\oplus\mathbb{Q}$ (Rasmussen considers rational coefficients).
- Recall that our vector spaces all possess a q-grading, so we can ask about the $\mathbb{Q}$-gradings of these copies of $\mathbb{Q}$.

- Theorem (Rasmussen, 2004): The q-gradings of these copies of $\mathbb{Q}$ are always odd and differ by 2.

- Definition: $s(K) := \frac{1}{2}$ (average of these gradings) $\in 2\mathbb{Z}$.
- Theorem (Rasmussen, 2004): $|s(K)| \le 2$ (slice genus of K)*
- Theorem (Rasmussen, 2004): s descends to a homomorphism

$$\mathrm{Conc}(S^3) \to \mathbb{Z},$$

where $\mathrm{Conc}(S^3)$ is the group of equivalence classes of knots in S^3 with the

* A knot is *slice* if it bounds a disk in the 4-ball. The slice genus of a knot is the minimal genus of a surface that it bounds in the 4-ball.

group operation # (connected sum), where two knots are equivalent if they differ by a slice knot under this operation.