Hyperbolic Geometry and Fuchsian Groups

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1 Constructing Hyperbolic Geometry

1.1 Introduction

The study of hyperbolic geometry was motivated by the denial of Euclid's Parallel Postulate, whose simplest statement is:

Axiom 1. Let L be a line and P be a point that does not lie on L. There exists precisely one line L' passing through P that never intersects L.

This axiom was long believed to be intuitively obvious, although some had unsuccessfully attempted to show that it followed from Euclid's other axioms. One such attempt was made by Giovanni Saccheri in a 1733 paper, in which he set out to show that the denial of Axiom 1 led to contradiction. Axiom 1 may be denied in one of two ways: one may assert that no such line L' exists, or one may assert that multiple distinct lines pass through P without intersecting L. The latter assertion serves as the foundation for hyperbolic geometry:

Axiom 2. Let L be a line and P be a point that does not lie on L. There exist multiple lines passing through P that never intersect L.

The preceding material is adapted from [4], which may be referenced for a more detailed treatment. It remains to provide an example of a geometry in which Axiom 2 holds. We will construct such a geometry by stipulating its group of isometries. We begin by defining the set on which we would like to construct our geometry.

Definition 3. The *upper half-plane* in $\mathbb{C}$ is the set $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

Let us now observe three types of transformations that map $\mathbb{H}$ to itself bijectively.

A *translation* is a function of the form $f(z) = z + a$ for some $a \in \mathbb{R}$. Translations simply shift the upper half-plane horizontally. A *scaling* is a function of the form $g(z) = az$ for some real $a > 0$. Scalings expand or contract points in $\mathbb{H}$ along lines emanating radially from the origin. Finally, we will call the function $h(z) = -\frac{1}{z}$ a *flip-inversion*. If we think of a point $z \in \mathbb{H}$ as a two-dimensional vector, then a flip-inversion inverts the magnitude of z and then flips the result over the imaginary axis.

We would like to define the length of a path in $\mathbb{H}$ such that any transformation that can be expressed as a composition of translations, scalings, and flip-inversions preserves distance. We will first demonstrate a succinct way of expressing such transformations that simultaneously illuminates the group structure of these transformations.

Theorem 4. *A mapping $\gamma : \mathbb{H} \to \mathbb{H}$ can be expressed as a composition of translations, scalings, and flip-inversions if and only if γ is of the form*

$$\gamma(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are real numbers such that $ad - bc > 0$.

We call a mapping of the above form a *Möbius transformation*, and we denote the set of Möbius transformations by Möb($\mathbb{H}$). We first prove the following lemma:

Lemma 5. Möb($\mathbb{H}$) *is preserved under composition of functions.*

Proof of Lemma 5. Let $\gamma(z) = \frac{az+b}{cz+d}$ and $\eta(z) = \frac{Az+B}{Cz+D}$ be Möbius transformations. Then

$$\begin{aligned} \eta \circ \gamma(z) &= \frac{A\frac{az+b}{cz+d} + B}{C\frac{az+b}{cz+d} + D} \\ &= \frac{\frac{Aaz+Ab+B(cz+d)}{cz+d}}{\frac{Caz+Cb+D(cz+d)}{cz+d}} \\ &= \frac{(Aa+Bc)z + (Ab+Bd)}{(Ca+Dc)z + (Cb+Dd)}. \end{aligned}$$

By assumption, we have $ad - bc > 0$ and $AD - BC > 0$. Therefore,

$$\begin{aligned} (Aa+Bc)(Cb+Dd) - (Ab+Bd)(Ca+Dc) &= ACab + BDcd + ADad + BCbc - ACab - BDcd - ADbc - BCad \\ &= ADad + BCbc - ADbc - BCad \\ &= (AD - BC)(ad - bc) \\ &> 0. \end{aligned}$$

Therefore $\eta \circ \gamma \in$ Möb($\mathbb{H}$). □

We now prove Theorem 4.

Proof of Theorem 4. We first show that any composition of translations, scalings, and flip-inversions is a Möbius transformation. By Lemma 5, it suffices to show that translations, scalings, and flip-inversions are themselves Möbius transformations. Consider a translation $f(z) = z + a$, $a \in \mathbb{R}$. We may write $f(z) = \frac{1z+a}{0z+1}$. Since $1 \cdot 1 - a \cdot 0 = 1 > 0$, the translation f must be a Möbius transformation. Consider a scaling $g(z) = az$, $a > 0$. We may write $g(z) = \frac{az+0}{0z+1}$. Since $a \cdot 1 - 0 \cdot 0 = a > 0$, the scaling g must be a Möbius transformation. Now consdier a flip-inversion $h(z) = -\frac{1}{z}$. We may write $h(z) = \frac{0z-1}{1z+0}$. Since $0 \cdot 0 - (-1) \cdot 1 = 1 > 0$, the flip-inversion h must be a Möbius transformation.

We now show that any Möbius transformation $\gamma(z) = \frac{az+b}{cz+d}$ is a composition of translations, scalings, and flip-inversions. First suppose that $c \neq 0$. Let

$$f_1(z) = z + \frac{d}{c}, \quad f_2(z) = -\frac{1}{z}, \quad f_3(z) = \frac{ad-bc}{c^2}z, \quad f_4(z) = z + \frac{a}{c}.$$

Clearly f_1 and f_4 are translations, f_2 is a flip-inversion, and f_3 is a scaling. We now have

$$\begin{aligned} f_4 \circ f_3 \circ f_2 \circ f_1(z) &= f_4 \circ f_3 \circ f_2 \left(z + \frac{d}{c} \right) \\ &= f_4 \circ f_3 \left(-\frac{1}{z + \frac{d}{c}} \right) \\ &= f_4 \left(\frac{bc - ad}{c^2 z + cd} \right) \\ &= \frac{bc - ad + (cz+d)a}{c^2 z + cd} \\ &= \frac{az+b}{cz+d} \\ &= \gamma(z). \end{aligned}$$

Now suppose $c = 0$. Then $\gamma(z) = \frac{a}{d}z + \frac{b}{d}$. Let $g_1(z) = \frac{a}{d}z$ and let $g_2(z) = z + \frac{b}{d}$. Since $ad - bc = ad > 0$, we see that g_1 is a scaling. Clearly g_2 is a translation, and $g_2 \circ g_1 = \gamma$. Therefore every Möbius transformation is a composition of translations, scalings, and flip-inversions. $\square$

There is even more structure to Möb($\mathbb{H}$) than Lemma 5 implies. Composition of functions is certainly associative, and if we take f_1 through f_4 as in the proof of Theorem 4, then a similar computation gives

$$\gamma^{-1}(z) = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_4^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Since $da - (-b)(-c) > 0$, we see that $\gamma^{-1} \in$ Möb($\mathbb{H}$). Finally, note that the identity map is a Möbius transformation, so we may conclude that Möb($\mathbb{H}$) is a group whose operation is composition of functions. Since every Möbius transformation is obviously continuous and has a continuous inverse, Möb($\mathbb{H}$) is a group of homeomorphisms from $\mathbb{H}$ to itself. This will be the group of isometries that we desire.

1.2 Hyperbolic Length and Volume

Let $\sigma : [a, b] \to \mathbb{H}$ be a differentiable function. We often identify σ with its image, calling σ a *path* in $\mathbb{H}$. We want to realize the length of σ as $\int_\sigma \rho = \int_a^b \rho(\sigma(t))|\sigma'(t)|dt$ for some positive continuous $\rho : \mathbb{H} \to \mathbb{R}$. It turns out that, up to multiplication by a constant, there is precisely one ρ such that the length of a path is invariant under Möbius transformations.

Theorem 6. *Let $\rho : \mathbb{H} \to \mathbb{R}$ be a positive continuous function. If $\int_{\gamma(\sigma)} \rho = \int_\sigma \rho$ for every path σ in $\mathbb{H}$ and every $\gamma \in$ Möb($\mathbb{H}$), then $\rho(z) = \frac{c}{\operatorname{Im}(z)}$ for some $c > 0$.*

Proof. This proof goes as outlined in [1]. Fix $\gamma \in$ Möb($\mathbb{H}$). By the change of variables theorem we have that $\int_{\gamma(\sigma)} \rho = \int_\sigma (\rho \circ \gamma)|\gamma'|$ for any path σ in $\mathbb{H}$. Since $\int_{\gamma(\sigma)} \rho = \int_\sigma \rho$, we have $\int_\sigma \rho = \int_\sigma (\rho \circ \gamma)|\gamma'|$, so that $\int_\sigma (\rho - (\rho \circ \gamma)|\gamma'|) = 0$ for every path σ. Suppose there is some $z_0 \in \mathbb{H}$ such that $(\rho - (\rho \circ \gamma)|\gamma'|)(z_0) \neq 0$. Then, since $\rho - (\rho \circ \gamma)|\gamma'|$ is continuous, there is some open neighborhood N of z_0 on which $\rho - (\rho \circ \gamma)|\gamma'|$ is nonzero. Then if σ is a path in N, we have $\int_\sigma (\rho - (\rho \circ \gamma)|\gamma'|) \neq 0$, a contradiction. Therefore $(\rho - (\rho \circ \gamma)|\gamma'|)(z) = 0$ for all $z \in \mathbb{H}$, so that

$$\rho = (\rho \circ \gamma)|\gamma'|. \tag{$\star$}$$

By ($\star$), we have $\rho(z) = \rho(z + b)$. If we fix z and let $b = -\operatorname{Re}(z)$, then we see that $\rho(z) = \rho(z - \operatorname{Re}(z)) = \rho(\operatorname{Im}(z)i)$. Therefore $\rho(z)$ only depends on $y = \operatorname{Im}(z)$, so there is a positive continuous function $f : (0, \infty) \to \mathbb{R}$ such that $\rho(z) = f(y)$. Again by ($\star$), we have $\rho(kz) = \rho(z)/k$, and therefore $f(ky) = f(y)/k$. If we fix $y = 1$, then we see that for any $k \in (0, \infty)$, we have $f(k) = f(1)/k$. If we set $c = f(1)$, then we may conclude that for any $z \in \mathbb{H}$, we have $\gamma(z) = f(\operatorname{Im}(z)) = \frac{c}{\operatorname{Im}(z)}$. $\square$

We will therefore define the *hyperbolic length* of a path σ to be $\int_\sigma \frac{1}{\operatorname{Im}(z)} = \int_a^b \frac{|\sigma'(t)|}{\operatorname{Im}(\sigma(t))}dt$. This informs us as to how we might define hyperbolic volume. Suppose we have some "very small" rectangle with a vertex at $z \in \mathbb{H}$ whose sides have Euclidean lengths dx and dy. Then, by our formulation of hyperbolic length, our rectangle's area should be $\frac{dx}{\operatorname{Im}(z)} \cdot \frac{dy}{\operatorname{Im}(z)} = \frac{1}{\operatorname{Im}(z)^2}dxdy$. Now let Ω be a region in $\mathbb{H}$. Then whenever $\int_\Omega \frac{1}{\operatorname{Im}(z)^2}dz$ is defined, we call this the *hyperbolic area* of Ω. As one might expect, hyperbolic volume is also invariant under Möbius transformations.

Theorem 7. *Let Ω be a region in $\mathbb{H}$ with hyperbolic area $A(\Omega) = \int_\Omega \frac{1}{\operatorname{Im}(z)}dz$. Then $A(\Omega) = A(\gamma\Omega)$ for every $\gamma \in$ Möb($\mathbb{H}$).*

Proof. The proof is outlined in Proposition 5.9.3 of [1]. $\square$

With a bit of work, it can be shown that the shortest path between any two points in $\mathbb{H}$ is either part of a circle centered on the real axis, or part of a vertical line. That is, circles centered on the real axis and vertical lines play the role of "lines" in $\mathbb{H}$. It can now be seen that $\mathbb{H}$ is a model for geometry that exhibits Axiom 2.

1.3 The Poincaré Disk

The geometry of $\mathbb{H}$ may also be represented in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Consider the map $h(z) = \frac{z-i}{iz-1}$. This map has the same form as a Möbius transformation, and so it is unsurprising that h is a continuous bijection. Rather than a bijection from $\mathbb{H}$ to itself, it is easily shown that h is a bijection from $\mathbb{H}$ to $\mathbb{D}$. We may then define the length of a path $\sigma : [0,1] \to \mathbb{D}$ simply by the formula $\int_{h^{-1}\sigma} \frac{1}{\text{Im}(h^{-1}(z))}$, and similarly for hyperbolic area. Therefore the geometry on $\mathbb{D}$ is the same as the geometry on $\mathbb{H}$. When $\mathbb{D}$ is equipped with these notions of length and area, we call $\mathbb{D}$ the *Poincaré Disk model* for hyperbolic geometry. The benefit of this model is that, since $\mathbb{D}$ is a compact subset of $\mathbb{C}$, the Poincaré Disk model allows us to easily draw the entire hyperbolic space on a piece of paper. We may talk about Möbius transformations of $\mathbb{D}$ by letting $\text{Möb}(\mathbb{D}) = \{h^{-1}\gamma h \mid \gamma \in \text{Möb}(\mathbb{H})\}$. Of course, $\text{Möb}(\mathbb{D})$ and $\text{Möb}(\mathbb{H})$ are isomorphic.

2 Group Actions

We have seen that the group $(\text{Möb}(\mathbb{H}), \circ)$ consists of elements that deform $\mathbb{H}$ in a way that reflects the group structure of $\text{Möb}(\mathbb{H})$. For $\gamma, \eta \in \text{Möb}(\mathbb{H})$ and $z \in \mathbb{H}$, we have $(\gamma \circ \eta)(z) = \gamma(\eta(z))$, and we also have that the identity element of $\text{Möb}(\mathbb{H})$ is the identity function on $\mathbb{H}$. For these reasons, we say that the group $\text{Möb}(\mathbb{H})$ *acts on* $\mathbb{H}$. Formally, we make the following definition.

Definition 8. Let G be a group and X be a set. A *group action* of G on X is a map $\Theta : G \times X \to X$, $(g, x) \mapsto g.x$ such that

(i) $g.(h.x) = (gh).x \quad \forall g, h \in G,\ x \in X$,

(ii) $e.x = x \quad \forall x \in X$,

where e is the identity element of G. For a subset Y of X we write $g.Y = \{g.y \mid y \in Y\}$.

Throughout, we will let G denote a group acting on a set X. We now establish some basic notions concerning group actions.

Definition 9. Let $x \in X$. Then $G_x = \{g \in G \mid g.x = x\}$ is called the *stabilizer group* of x, and $G(x) = \{g.x \in X \mid g \in G\}$ is called the *orbit* of x. If $g.x = x$, we say that x is *fixed* by g.

We call a group action *effective* if the identity is the only group element that fixes every $x \in X$, and we call a group action *free* if every $x \in X$ has a trivial stabilizer group. Clearly every free action is effective. We now give examples of subgroups of $\text{Möb}(\mathbb{H})$ that demonstrate these definitions; in particular, we see that there exist actions that are effective but not free.

Example 10. Let $G = \{z \mapsto z + k \mid k \in \mathbb{Z}\} \subseteq \text{Möb}(\mathbb{H})$. Then G acts freely on $\mathbb{H}$ because for any $z \in \mathbb{H}$, $z + k \neq z$ whenever $k \neq 0$. Now consider the group $\mathbb{Z}/6\mathbb{Z} = \{[x] \subseteq \mathbb{Z} \mid y \in [x] \iff y \equiv x \pmod 6\}$ with addition $[x] + [y] = [x + y]$. Then $\mathbb{Z}$ acts on $\mathbb{Z}/6\mathbb{Z}$ by addition: $k.[x] = [k + x]$. This action is not effective because, for any $[x] \in \mathbb{Z}/6\mathbb{Z}$, we have $6.[x] = [6{+}x] = [x]$. Finally, consider $H = \{z \mapsto -\frac{1}{z}, z \mapsto z\} \subseteq \text{Möb}(\mathbb{H})$. H acts effectively on $\mathbb{H}$, because $z \mapsto -\frac{1}{z}$ does not fix every point. However, H does not act freely on $\mathbb{H}$, because $-\frac{1}{i} = i$, so i has a nontrivial stabilizer group.

When X is a topological space, we make the following definition.

Definition 11. Let X/G be the space of equivalence classes of the relation $x \sim y \iff x \in G(y)$ endowed with the quotient topology. We call X/G the *orbit space* of X.

We often also endow G with a topology, and we say that a *discrete group* is a group endowed with the discrete topology. The discrete groups we consider will usually arise as discrete subspaces of groups with more interesting topology. If H is a subgroup of G, and the subspace topology on H inherited from the topology on G is discrete, then H is called a *discrete subgroup* of G.

Example 12. The set $\mathbb{R}$ of real numbers with its usual topology is a group with respect to addition. The subgroup $\mathbb{Z} \subset \mathbb{R}$ is discrete, since every integer may be encompassed by an open interval containing no other integer. Furthermore, $\mathbb{Z}$ acts on $\mathbb{R}$ by addition, and so the orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1 via the homeomorphism $[x] \mapsto (\cos(2\pi x), \sin(2\pi x))$.

We make the following definition concerning group actions.

Definition 13. Let G be a discrete group acting on a topological space X. We call the action of G on X *properly discontinuous* if the following conditions hold:

(i) For every $x \in X$, there is a neighborhood U of x such that $(g.U) \cap U = \varnothing$ for all but finitely many $g \in G$.

(ii) If $x, y \in X$ such that $x \notin G(y)$, then there are neighborhoods U of x and V of y such that $(g.U) \cap V = \varnothing$ for all $g \in G$.

The action of $\mathbb{Z}$ on $\mathbb{R}$ as in Example 12 is properly discontinuous, as can be seen by considering open neighborhoods of radius 1 for condition (i), and for condition (ii) letting $U = (x - \frac{1}{2}\delta, x + \frac{1}{2}\delta)$, $V = (y - \frac{1}{2}\delta, y + \frac{1}{2}\delta)$ where $\delta = \min\{|kx - y| \mid k \in \mathbb{Z}\}$. In Example 18 we will see a discrete group that does not act properly discontinuously.

An equivalent definition of properly discontinuous group actions is given in [1]:

Definition 13 (Equivalent). Let G be a discrete group acting on a topological space X. Then G acts *properly discontinuously* on X if, for every $x \in X$ and every nonempty compact $K \subseteq X$, the set $\{g \in G \mid g.x \in K\}$ is finite.

3 Fuchsian Groups

Fuchsian groups are discrete subgroups of Möb($\mathbb{H}$). But with respect to what topology on Möb($\mathbb{H}$)? We will see that Möb($\mathbb{H}$) admits of a very natural topology, and we first need the following lemma.

Lemma 14. *If γ is a Möbius transformation, then there exist $a, b, c, d \in \mathbb{R}$ such that*

$$\gamma(z) = \frac{az + b}{cz + d},$$

where $ad - bc = 1$. We call $\frac{az+b}{cz+d}$ a normalized form *for γ.*

Proof. We already know that $\gamma(z) = \frac{az+b}{cz+d}$ where $ad - bc > 0$. Now let $\lambda = (ad - bc)^{-1/2}$. Then $\gamma(z) = 1 \cdot \frac{az+b}{cz+d} = \frac{\lambda}{\lambda} \cdot \frac{az+b}{cz+d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}$, and $(\lambda a)(\lambda d) - (\lambda b)(\lambda c) = 1$. □

It remains to see whether these a, b, c, d are unique. Indeed, they are unique so long as we adopt the convention that a be nonnegative.

Lemma 15. *For any Möbius transformation γ, there exists precisely one 4-tuple $(a, b, c, d) \in \mathbb{R}^4$ such that $a \geq 0$ and $\frac{az+b}{cz+d}$ is a normalized form for γ.*

Proof. Suppose $(a, b, c, d), (A, B, C, D) \in \mathbb{R}^4$, $a, A \geq 0$, and $\frac{az+b}{cz+d}, \frac{Az+B}{Cz+D}$ are normalized forms for γ. Since $\gamma^{-1}(z) = \frac{Dz-B}{-Cz+A}$, we have

$$\begin{aligned} z &= \frac{D\left(\frac{az+b}{cz+d}\right) - B}{-C\left(\frac{az+b}{cz+d}\right) + A} \\ &= \frac{(Da - Bc)z + (Db - Bd)}{(-Ca + Ac)z + (-Cb + Ad)} \end{aligned}$$

Since $(Da - Bc)(-Cb + Ad) - (Db - Bd)(-Ca + Ac) = (AD - BC)(ad - bc) = 1$, the above must be a normalized form for z. Therefore we have the system of equations

$$\begin{aligned} Da - Bc &= \pm 1 \\ Db - Bd &= 0 \\ Ac - Ca &= 0 \\ Ad - Cb &= \pm 1. \end{aligned}$$

Solving this system gives us

$$\begin{aligned} a &= \pm A \\ b &= \pm B \\ c &= \pm C \\ d &= \pm D. \end{aligned}$$

Since $a, A \geq 0$, we must choose the positive sign. Therefore $(a, b, c, d) = (A, B, C, D)$. □

We are now in the position to define a metric on Möb($\mathbb{H}$). Let φ : Möb($\mathbb{H}$) $\to \mathbb{R}^4$ be the map that takes γ to the unique tuple (a, b, c, d) given by Lemma 15. Then for $\gamma, \eta \in$ Möb($\mathbb{H}$) we define the metric $d(\gamma, \eta) = \|\varphi(\gamma) - \varphi(\eta)\|$, where $\|\cdot\|$ is the standard Euclidean norm. This metric induces a topology on Möb($\mathbb{H}$), and our definition of a Fuchsian group now makes sense.

Definition 16. A Fuchsian group is a discrete subgroup of Möb($\mathbb{H}$).

We have already seen an example of a Fuchsian group: the group G in Example 10. We now note some equivalent conditions for a subgroup Γ of Möb($\mathbb{H}$) to be a Fuchsian group. We make use of results in [1] when they are outside the scope of this paper.

Theorem 17. *Let* Γ *be a subgroup of* Möb($\mathbb{H}$). *The following are equivalent:*

(i) Γ *is a Fuchsian group.*

(ii) The identity element of Γ *is isolated.*

(iii) For each $z \in \mathbb{H}$*, the orbit* $\Gamma(z)$ *is a discrete subset of* $\mathbb{H}$.

(iv) Γ *acts properly discontinuously on* $\mathbb{H}$.

Proof. We go in three steps. In Step 1, we show that (i) $\iff$ (ii). In Step 2, we show that (iii) $\iff$ (iv). Finally, in Step 3, we show that (i) $\iff$ (iv). Step 3 is due to [1], and Step 2 is adapted from [1].

Step 1. That (i) $\implies$ (ii) is obvious. To show that (ii) $\implies$ (i), observe that the map $\eta \mapsto \gamma \circ \eta$ is a continuous map on Γ. Therefore, if $\{\eta_n\}$ is a sequence of distinct elements of Γ such that $\eta_n \to \gamma$, then $\gamma^{-1}\eta_n \to \gamma^{-1}\gamma = e$, where e is the identity. It follows that whenever (ii) holds, (i) must also hold. We conclude that (i) $\iff$ (ii).

Step 2. We first show that (iv) $\implies$ (iii). Suppose that there is some $z \in \mathbb{H}$ such that $\Gamma(z)$ is not discrete. Then there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ and some $z_0 \in \mathbb{H}$ such that $\gamma_n(x) \to z_0$. Then for any $\varepsilon > 0$ there are infinitely many $\gamma_n \in \Gamma$ such that $\gamma_n(z)$ is the the closed ball K of radius ε about z_0. Since K is compact, this contradicts (iv), so we see that (iii) $\implies$ (iv).

Now we show that (iii) $\implies$ (iv). Suppose that $\Gamma(z)$ is discrete for every $z \in \mathbb{H}$. Since Möb($\mathbb{H}$) $\cong$ Möb($\mathbb{D}$), we may consider Γ as a subgroup of Möb($\mathbb{D}$) acting on $\mathbb{D}$. We first show that the stabilizer group of every $z_0 \in \mathbb{D}$ is finite. By S_{z_0} denote the stabilizer group of z_0. Let $\gamma \in S_{z_0}$ be other than the identity, so that γ has the fixed point z_0 in $\mathbb{D}$. Such Möbius transformations are called elliptic, and elliptic transformations are conjugate to rotations $R_\theta : z \mapsto e^{i\theta}z$, $\theta \in [0, 1]$ by Proposition 11.3.1 in [1]. That is, $S_{z_0} = \eta^{-1}T\eta$ for some $\eta \in$ Möb($\mathbb{D}$), where $T = \{R_\theta \mid \theta \in [0, 1]\}$. If S_{z_0} were infinite, there would exist infinitely many distinct $\theta_j \in [0,1]$ such that R_θ is conjugate to an element of S_{z_0}. By the compactness of $[0,1]$, there exists a subsequence $\{\theta_{j_n}\}$ that converges to some $\varphi \in [0,1]$. Then, since S_{z_0} is a group, the elements $R_{-\theta_{j_n}}$ are also conjugate to elements of S_{z_0}. Therefore the sequence $\{\rho_n = R_{\theta_{j_n} - \theta_{j_{n+1}}}\}$ is also a sequence of elements conjugate to members of S_{z_0}, and this sequence clearly converges to R_0, which is conjugate to the identity (indeed, it is the identity). Now, let z_1 be an element of $\mathbb{D}$ such that $\eta z_1 \neq 0$. Then $\rho_n(\eta z_1) \to \eta z_1$. If we let $\gamma_n = \eta^{-1}\rho_n\eta \in S_{z_0}$, then we see that $\gamma_n(z_1) = \eta^{-1}\rho_n(\eta z_1) \to z_1$, and so the orbit of z_1 under S_{z_0} is not discrete, contradicting our assumption. Therefore every element of $\mathbb{D}$ has a finite stabilizer group.

Now we will show that Γ acts properly discontinuously. Let K be a compact subset of $\mathbb{H}$, and let $z \in \mathbb{H}$. We want to show that $\Sigma = \{\gamma \in \Gamma \mid \gamma(z) \in K\}$ is finite. Suppose that Σ is infinite, so that there is an infinite sequence $\{\gamma_n\}$ of elements of Σ. Consider $\{\gamma_n(x)\} \subseteq K$. If $\{\gamma_n(z)\}$ is infinite, then by the compactness of K, the sequence $\{\gamma_n(z)\}$ has a convergent subsequence, and we can assume that all points in this subsequence are distinct. Therefore $\{\gamma_n(z)\} \subseteq \Gamma(z)$ has a limit point, contradicting the fact that $\Gamma(z)$ is discrete. If $\{\gamma_n(z)\}$ is finite, then we can write $\{\gamma_n(z)\} = \{z^1, \dots, z^k\}$, and then partition $\{\gamma_n\} \subseteq \Sigma$ into subsets $\Sigma_j = \{\gamma_n \mid \gamma_n(z) = z^j\}$. We want to show that each Σ_j is finite. By the preceding paragraph we know that the stabilizer group $S_z = \{g_1, \dots, g_r\}$ is finite. For each j, pick some $\eta^j \in \Sigma_j$. Then for any $\gamma \in \Sigma_j$, we have $(\eta^j)^{-1}\gamma \in S_z$, and so there is some ℓ such that $(\eta^j)^{-1}\gamma = g_\ell$, and so $\gamma = \eta^j g_\ell$. Therefore there are only finitely many possibilities for γ, and so Σ_j is finite. Since the sets Σ_j partition Σ, we conclude that Σ is finite. Therefore in must be the case that Γ acts properly discontinuously.

Step 3. By Definition 13, all groups that act properly discontinuously are discrete, and so (iv) $\implies$ (i). Now we show that (i) $\implies$ (iv). Suppose Γ satisfies every condition of Definition 13 except for discreteness. By Proposition 24.2.3 in [1], there exists some $z \in \mathbb{H}$ such that the only element of Γ that fixes z is the identity. Since Γ is not discrete, Step 1 tells us that the identity e is not isolated, and so there is some sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n \to e$, and so $\gamma_n(z) \to z$, while $\gamma_n(z) \neq z$ for any n. Now let U be some neigbhorhood of z. Then $\gamma_n(z) \in U$ for large enough n. That $\gamma_n(z) \in \gamma_n.U$ is trivial. Therefore $(\gamma_n.U) \cap U \neq \varnothing$ for infinitely many n, contradicting the first condition of Definition 13. Therefore (i) $\implies$ (iv). □

It is often helpful to consider the boundary $\partial\mathbb{H}$ of $\mathbb{H}$, which is taken to be the real axis along with the "point at infinity" ∞. We must be careful when we do this, because Möbius transformations behave differently on $\partial\mathbb{H}$ than they do on $\mathbb{H}$.

Example 18. Consider the Fuchsian group $K = \{z \mapsto 2^k z \mid k \in \mathbb{Z}\}$. By Theorem 17, K acts properly discontinuously on $\mathbb{H}$. The action of K also extends to $\mathbb{H} \cup \partial\mathbb{H}$. However, this action is no longer properly discontinuous, because $0 \in \gamma.U$ for every open neighborhood U of 0 and every $\gamma \in K$.

4 Fundamental Domains

Fuchsian groups act on $\mathbb{H}$ in such a way that tessellate certain figures in the upper half-plane, just as the group $\mathbb{Z} \times \mathbb{Z}$ tessellates the unit square in $\mathbb{R}^2$. Given a Fuchsian group Γ, any figure F that is tessellated by Γ is called a *fundamental domain* for Γ.

Definition 19. Let Γ be a Fuchsian group, and let F be an open subset of $\mathbb{H}$. Let $\overline{F}$ denote the closure of F (i.e. the union of F with its set of limit points). We say that F is a *fundamental domain* if $\bigcup_{\gamma\in\Gamma} \gamma.\overline{F} = \mathbb{H}$ and $\gamma_1.F \cap \gamma_2.F = \varnothing$ for any $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

We have already seen some examples of Fuchsian groups, and so now we look at their fundamental domains. Recall the group $G = \{z \mapsto z + k \mid k \in \mathbb{Z}\}$ from Example 10.Then the sets $I_k = \{z \in \mathbb{C} \mid k < \mathrm{Re}(z) < k+1\}$ are fundamental domains for G.

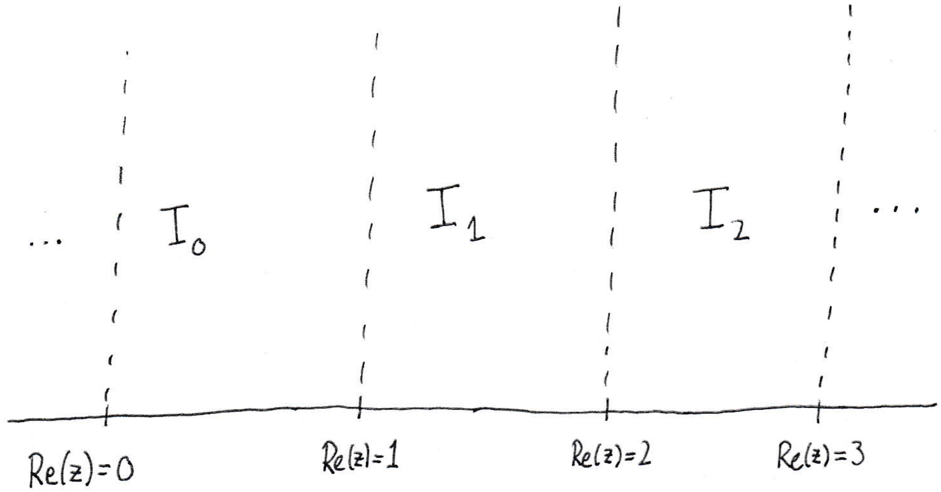

Figure 1. Fundamental Domains for G

Now recall the group K from Example 18. Then the sets $D_k = \{z \in \mathbb{C} \mid 2^k < |z| < 2^{k+1}\}$ are fundamental domains for K.

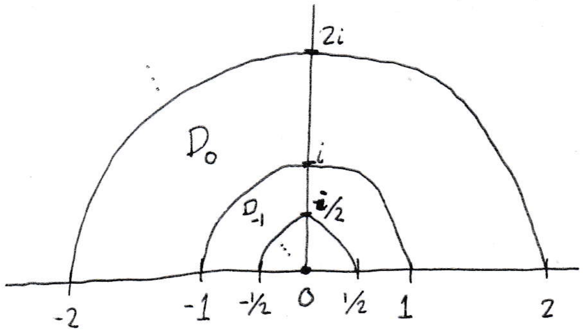

Figure 2. Fundamental Domains for K

Finally, we will see that we can associate to a Fuchsian group Γ a certain constant: the area of any fundamental domain of Γ. That is, Γ determines the area of any finite-area fundamental domain of Γ. Let $A(S)$ denote the hyperbolic area of a set $S \subseteq \mathbb{H}$, and let us only consider open sets S where $A(\overline{S}) = A(S)$. The following proof is adapted from [1].

Theorem 20. *Let F_1, F_2 be fundamental domains for a Fuchsian group Γ with finite area. Then $A(F_1) = A(F_2)$.*

Proof. We see that $\overline{F}_1 \supseteq \bigcup_{\gamma\in\Gamma}(\overline{F}_1 \cap \gamma.F_2)$. Since F_2 is a fundamental domain, the sets $\overline{F}_1 \cap \gamma.F_2$ are all disjoint, and so the area of their union is the union of their areas. Therefore we have $A(\overline{F}_1) \geq \sum_{\gamma\in\Gamma} A\left(\overline{F}_1 \cap \gamma.F_2\right)$. Since each γ is a bijection, we have $\overline{F}_1 \cap \gamma.F_2 = \gamma.(\gamma^{-1}.\overline{F}_1 \cap F_2)$. Since Möbius transformations are area-preserving, we have $A(\gamma.(\gamma^{-1}.\overline{F}_1 \cap F_2)) = A(\gamma^{-1}.\overline{F}_1 \cap F_2))$. Since Γ is a group, every element of Γ is the inverse of some other element, and so

$$A(\overline{F}_1) \geq \sum_{\gamma\in\Gamma} A(\gamma^{-1}.\overline{F}_1 \cap F_2)) = \sum_{\gamma\in\Gamma} A(\gamma.\overline{F}_1 \cap F_2)),$$

Since F_1 is a fundamental domain, we have $\bigcup_{\gamma\in\Gamma} \gamma.\overline{F}_1 = \mathbb{H}$. Thus

$$A(\overline{F}_1) \geq \sum_{\gamma\in\Gamma} A\left(\gamma.\overline{F}_1 \cap F_2\right) \geq A\left(\bigcup_{\gamma\in\Gamma} \gamma.\overline{F}_1 \cap F_2\right) = A(F_2).$$

Interchanging the roles of F_1 and F_2, our result follows. □

Now that we have developed the basic theory of fundamental domains, it is possible to explore the rich world of tessellating $\mathbb{H}$. While the plane only admits of finitely many tessellations by convex polygons, it turns out that $\mathbb{H}$ admits of infinitely many! There is much more to be said on the topic of Fuchsian groups, it is hoped that this paper may serve as a bridge to this topic.

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