A Unified Approach to Fiedler-Like Pencils

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Matrix Polynomials

Definition: Matrix Polynomials

A matrix polynomial $P(\lambda)$ is a matrix whose entries are polynomials over a field $\mathbb{F}$, or equivalently, is a polynomial whose coefficients are matrices over $\mathbb{F}$.

$$\begin{bmatrix} 2\lambda^2 + 17 & \lambda \\ -\lambda & 17 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}.$$

Definition: Eigenvalues

Given a matrix polynomial $P(\lambda) = A_k\lambda^k + \cdots + A_0$, an element $\lambda_0 \in \overline{\mathbb{F}}$ is a finite eigenvalue of P if $\operatorname{rank} P(\lambda_0) < \operatorname{rank} P(\lambda)$. If $\operatorname{rank} A_k < \operatorname{rank} P(\lambda)$, then $P(\lambda)$ is said to have an eigenvalue at infinity.

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Motivation

Definition: Complete Polynomial Eigenproblem

- The eigenstructure of $P(\lambda)$ consists of all the eigenvalues along with data about their multiplicities.
- The singular structure of $P(\lambda)$ contains information about degrees of polynomial bases of the nullspaces of $P(\lambda)$.

The calculation of these structures for some $P(\lambda)$ is called the Complete Polynomial Eigenproblem (CPE).

Many engineering applications demand accurate solutions to the CPE. In particular, this problem arises from systems of ODEs, and from discretizations of PDEs, that are considered in the analysis of vibrations.

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Linearizations

Definition: Linearization

- A linearization of $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a matrix pencil $\mathcal{L}(\lambda)$ that shares its finite eigenstructure.
- A strong linearization of $P(\lambda)$ shares its eigenvalue at infinity.

Good linearizations:

- are simple to construct and have a block structure that is a template
- have good numerical properties: conditioning, backward errors
- preserve the structure of $P(\lambda)$ (symmetric, unitary, etc.)

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Fiedler Pencils

Definition: Fiedler pencil (Fiedler, 2003; Antoniou, Vologiannidis, 2004)

A Fiedler pencil for $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ is a matrix

$$F(\lambda) = M_{-k}\lambda - M_q \in \mathbb{F}[\lambda]^{nk \times nk},$$

where $M_q = M_{q(0)} M_{q(1)} \cdots M_{q(k-1)}$ and e.g. M_i for $0 < i < k$ is given by:

$$M_i := \left[\begin{array}{c|cc|c} I_{n(k-i-1)} & & & \\ \hline & -A_i & I_n & \\ & I_n & 0 & \\ \hline & & & I_{n(i-1)} \end{array}\right] \qquad \text{(elementary matrices)}$$

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Fiedler Pencils

Example:

Let $P(\lambda)$ be of degree 5 and $q = (2, 3, 4, 0, 1)$. Then the Fiedler pencil is:

$$\begin{bmatrix} \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ A_3 & \lambda I_n & -I_n & 0 & 0 \\ A_2 & 0 & \lambda I_n & A_1 & -I_n \\ -I_n & 0 & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & A_0 & \lambda I_n \end{bmatrix}$$

Generalized Fiedler Pencils with Repetition

Definition: GFPR (Bueno, Dopico, Furtado, Rychnovsky, 2015)

Given a square matrix polynomial $P(\lambda)$, a GFPR for $P(\lambda)$ is a pencil of the form

$$M_{\ell_q}(\mathcal{X})M_{\ell_z}(\mathcal{Y})(\lambda M_z - M_q)M_{r_q}(\mathcal{Z})M_{r_z}(\mathcal{W}),$$

where e.g. $M_{\mathbf{r}_q}(\mathcal{Z}) = M_{\mathbf{r}_q(0)}(Z_0)M_{\mathbf{r}_q(1)}(Z_1)\cdots M_{\mathbf{r}_q(m)}(Z_m)$ and $M_i(Z_j)$ for $0 < i < k$ is given by

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A Canonical Form for Fiedler Pencils

Theorem (Dopico, Lawrence, Pérez, Van Dooren, 2016)

If $F(\lambda)$ is a Fiedler pencil for the matrix polynomial $P(\lambda)$, then through block row and column permutations, $F(\lambda)$ can be expressed in the canonical form

$$F(\lambda) \rightsquigarrow \left[\begin{array}{c|c} M & K_2^T \\ \hline K_1 & 0 \end{array}\right], \qquad \text{(BMBP)}$$

where K_i for $i = 1, 2$ is of the form

$$\begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{F}[\lambda]^{p_i n \times (p_i+1)n},$$

and M has a staircase-shaped pattern with the blocks

$$\lambda A_k + A_{k-1}, A_{k-2}, \ldots, A_0.$$

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Λ-dual Pencils

- The K_i blocks are "dual" to $\begin{bmatrix} \lambda^{p_i} & \lambda^{p_i-1} & \cdots & 1 \end{bmatrix}$ in the sense that

$$\begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \cdot \begin{bmatrix} \lambda^{p_i} \\ \lambda^{p_i-1} \\ \vdots \\ 1 \end{bmatrix} = 0.$$

- The M block allows recovery of $P(\lambda)$:

$$\begin{bmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda A_5 + A_4 & 0 \\ A_3 & 0 \\ A_2 & A_1 \\ 0 & A_0 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = A_5\lambda^5 + \cdots + A_1\lambda + A_0.$$

- We say that a pencil of the form

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with these duality and recovery conditions is a Λ-dual pencil for $P(\lambda)$.

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A Λ-dual pencil

$$\left[\begin{array}{c|c} M & K_2^T \\ \hline K_1 & 0 \end{array}\right]$$

for $P(\lambda)$ is a strong linearization of $P(\lambda)$ if

- The linear coefficient matrices of $K_1(\lambda)$ and $K_2(\lambda)$ have full rank.
- For every $\lambda_0 \in \overline{\mathbb{F}}$, the matrices $K_1(\lambda_0)$ and $K_2(\lambda_0)$ have full rank.

A Canonical Form for GFPRs

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$$G(\lambda) \rightsquigarrow \left[\begin{array}{c|c} M & K_2^T \\ \hline K_1 & 0 \end{array}\right], \qquad (\Lambda\text{-Dual Pencil})$$

where for certain p_1, p_2,

$$K_i \cdot \begin{bmatrix} \lambda^{p_i} & \lambda^{p_i - 1} & \cdots & 1 \end{bmatrix}^T = 0, \quad i = 1, 2,$$

and

$$\begin{bmatrix} \lambda^{p_2} & \lambda^{p_2 - 1} & \cdots & 1 \end{bmatrix} \cdot M \cdot \begin{bmatrix} \lambda^{p_1} & \lambda^{p_1 - 1} & \cdots & 1 \end{bmatrix}^T = P(\lambda).$$

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Example: Canonical form for $(\lambda M_{-5} - M_{2,3,4,0,1})M_3(Z)$

$$\begin{bmatrix} \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ A_3 & \lambda I_n & -I_n & 0 & 0 \\ A_2 & 0 & \lambda I_n & A_1 & -I_n \\ -I_n & 0 & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & A_0 & \lambda I_n \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & Z & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & I_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda A_5 + A_4 & -Z & -I_n & 0 & 0 \\ A_3 & \lambda Z - I_n & \lambda I_n & 0 & 0 \\ A_2 & \lambda I_n & 0 & A_1 & -I_n \\ -I_n & 0 & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & A_0 & \lambda I_n \end{bmatrix}$$

Permute $\rightsquigarrow$

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Conclusions and Future Work

- Unification of Fiedler-like pencils under a canonical form. (Goodbye GFPRs)
- A unified search for structure-preserving linearizations.
- Analysis of numerical properties of this larger class of pencils
- Linearizations for matrix polynomials in nonmonomial bases

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  Monomial Basis: $1, \ \lambda, \ \lambda^2, \ \lambda^3, \ \ldots$
  Chebyshev Basis: $1, \ 2\lambda, \ 4\lambda^2-1, \ 8\lambda^3-4\lambda, \ \ldots$

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