Constructing e using first-year calculus

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In these 3-page notes, we prove the following facts, assuming only a background in the first two semesters of a standard course on calculus:

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$
- $\frac{d}{dx}e^x = e^x$
- $e = \sum_{n=0}^{\infty} \frac{1}{n!}$
- $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

1 Only logarithms convert multiplication to addition

We begin by investigating continuous functions $f(x)$ that satisfy the equation

$$f(xy) = f(x) + f(y). \tag{1}$$

Suppose that there is some $b > 0$ such that $f(b) \neq 0$. For any rational number p/q, we can assume that p is positive by letting q be negative whenever p/q is negative. Then we have

$$\begin{aligned} f(b^{p/q}) &= f(\underbrace{b^{1/q}b^{1/q}\cdots b^{p/q}}_{p \text{ times}}) \\ &= \underbrace{f(b^{1/q}) + f(b^{1/q}) + \cdots + f(b^{1/q})}_{p \text{ times}} \\ &= pf(b^{1/q}). \end{aligned}$$

In particular, when p is equal to q, we get $f(b) = qf(b^{1/q})$, and so $f(b^{1/q}) = (1/q)f(b)$. In conclusion, we have

$$f(b^{p/q}) = (p/q)f(b).$$

Because we are assuming the function f is continuous, and because every real number x is a limit of rational numbers[1] , we have

$$f(b^x) = f(b^{\lim r}) = f(\lim b^r) = \lim f(b^r) = \lim rf(b) = xf(b),$$

where lim stands for $\lim\limits_{\substack{r\to x\\ r\in\mathbb{Q}}}$. In particular, we have $f(b^{1/f(b)}) = (1/f(b))f(b) = 1$. Let us define $a = b^{1/f(b)}$, so that $f(a) = 1$.

[1] Every real number has a decimal expansion $x = a_0.a_1a_2a_3\ldots$, and every finite decimal is a rational number, so $x = \lim_{n\to\infty} a_0.a_1a_2\ldots a_n$.

Since $f(a) \neq 0$, all of the above formulas are true with a in place of b. Therefore $f(a^x) = x$. Also, $f(-1) = 0$, because $f(1) = f(a^0) = 0$, and

$$\begin{aligned} f(1) &= f((-1) \cdot (-1)) \\ 0 &= f(-1) + f(-1) = 2f(-1). \end{aligned}$$

Therefore $f(-a^x) = f((-1) \cdot a^x) = 0 + f(a^x) = x$. Every $x > 0$ is eqaul to $a^{\log_a(x)}$, and every $x < 0$ is equal to $-a^{\log_a(-x)}$, and so we conclude that, for $x \neq 0$, we have

$$f(x) = \log_a |x|.$$

Since $\lim_{x \to 0} |f(x)| = \infty$, we see that f cannot have an output value at $x = 0$ while still being continuous. In conclusion, we have the following theorem.

Theorem 1.1. *Let $f(x)$ be a continuous function satisfying Equation 1. Then there exists $a > 0$ such that*

$$f(x) = \log_a |x|.$$

2 The integral of $\frac{1}{t}$ is a logarithm

Consider the function $f(x) = \int_1^x \frac{1}{t} dt$ for $x > 0$. For $y > 0$, we have

$$f(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt.$$

Using the change of variables $u = xt$, we get

$$\int_x^{xy} \frac{1}{t} dt = \int_1^y \frac{1}{u/x} (1/x) du = \int_1^y \frac{1}{u} du.$$

Therefore,

$$f(xy) = \int_1^x \frac{1}{t} dt + \int_1^y \frac{1}{u} du = f(x) + f(y),$$

which means that the function $f(x)$ satisfies Equation 1. Since $f(x)$ is the integral of a continuous function, it is continuous, and so by Theorem 1.1, there is some number e such that $f(x) = \log_e |x|$. Since we are only considering $x > 0$, we may drop the absolute value symbol, and write $f(x) = \log_e(x)$. We write $\ln(x)$ instead of $\log_e(x)$.

3 The derivative of an exponential function

Let us continue to write $f(x) = \ln(x) = \int_1^x \frac{1}{t} dt$. By the Fundamental Theorem of Calculus, we have $\frac{d}{dx} f(x) = 1/x$. Since $f(e^x) = x$, we have $f^{-1}(x) = e^x$. By the formula for the derivative of an inverse function, we have

$$\frac{d}{dx} e^x = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{1/e^x} = e^x.$$

For any $a > 0$, we have $a^x = e^{\ln(a)x}$. By the Chain Rule, we have $\frac{d}{dx} a^x = \ln(a) e^{\ln(a)x} = \ln(a) a^x$.

4 A series formula for e

Let $g(x) = e^x$. Since $g'(x) = e^x = g(x)$, the nth derivative $g^{(n)}(x)$ is also equal to $g(x) = e^x$, and so $g^{(n)}(0) = e^0 = 1$. Therefore the Taylor series for e^x about $x = 0$ is $\sum_{n=0}^{\infty} g^{(n)}(0)\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We show that this series converges for every $x \in \mathbb{R}$.

We first show that the geometric series $\sum_{n=0}^{\infty} k^{-n}$ converges to $\frac{k}{k-1}$ for every number $k > 1$. Observe that

$$\begin{aligned}(1-x)(1+x+x^2+\cdots+x^N) &= 1-x+x-x^2+\cdots-x^N+x^N-x^{N+1}\\ (1-x)(1+x+x^2+\cdots+x^N) &= 1-x^{N+1}\\ 1+x+x^2+\cdots+x^N &= \frac{1-x^{N+1}}{1-x}\end{aligned}$$

Letting $x = k^{-1}$, this becomes $\sum_{n=0}^{N} k^{-n} = \frac{1-k^{-(N+1)}}{1-k^{-1}}$, and so

$$\sum_{n=0}^{\infty} k^{-n} = \lim_{N\to\infty}\sum_{n=0}^{N} k^{-n} = \lim_{N\to\infty}\frac{1-k^{-(N+1)}}{1-k^{-1}} = \frac{1}{1-k^{-1}} = \frac{k}{k-1}.$$

Let k be an integer greater than $|x|$. Then $\frac{|x|^n}{n!} < \frac{k^n}{n!}$, and so by the Direct Comparison Test, it suffices to show that $\sum_{n=0}^{\infty} \frac{k^n}{n!}$ converges for every integer $k > 1$. By the Direct Comparison Test and the fact that $\sum_{n=0}^{\infty} k^{-n}$ converges, it suffices to show that for all sufficiently large n, we have $\frac{k^n}{n!} < k^{-n}$, or equivalently, $k^{2n} < n!$.

When n is greater than k^3, the number $n!$ is greater than $A_n = (k^3)! \cdot k^{3(n-k^3)}$, which is divisible by k at least $3(n-k^3)$ times. But k^{2n} is divisible by k exactly $2n$ times. The difference $3(n-k^3) - 2n = n - 3k^3$ is positive for sufficiently large n, since $\lim_{n\to\infty}(n - 3k^3) = \infty$, and so A_n is eventually divisible by k more times than k^{2n} is. That means A_n is eventually larger than k^{2n}. Since $n! > A_n$ when $n > k^3$, we are done.

We conclude that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every $x \in \mathbb{R}$, and so the formula $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is true everywhere. In particular, we have

$$e = e^1 = \sum_{n=0}^{\infty} \tfrac{1}{n!}.$$

5 A limit formula for e

We show that $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$. Since $\ln(x)$ is an increasing continuous function with no horizontal asymptotes, it suffices to show that $\lim_{n\to\infty} \ln((1 + \frac{x}{n})^n) = x$. Letting $h = \frac{x}{n}$, we have

$$\begin{aligned}\lim_{n\to\infty} \ln\left((1+\tfrac{x}{n})^n\right) &= \lim_{h\to 0} \ln\left((1+h)^{\frac{x}{h}}\right)\\ &= \lim_{h\to 0} x\frac{\ln(1+h)}{h}\\ &= x\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}\\ &= x\left.\frac{d}{dt}\right|_{t=1}\ln(t)\\ &= x\frac{1}{1}\\ &= x.\end{aligned}$$

In particular, we have $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.