Principal Bundles

Day 5: Vector Bundles

Bradley Zykoski

June 19, 2020

Course summary

Today is the last day! Let's summarize what we've learned so far.

Course summary

Today is the last day! Let's summarize what we've learned so far.

1. A principal G-bundle is a fiber bundle $\pi : P \to B$ with a right G-action $P \curvearrowright G$ so that the fibers of P are G-torsors.

Course summary

Today is the last day! Let's summarize what we've learned so far.

1. A principal G-bundle is a fiber bundle $\pi : P \to B$ with a right G-action $P \curvearrowright G$ so that the fibers of P are G-torsors.
2. A principal connection is a G-equivariant splitting
$$0 \longrightarrow VP \underset{\overset{\omega}{\dashleftarrow}}{\longrightarrow} TP \xrightarrow{d\pi} \pi^* TB \longrightarrow 0,$$
and since $VP \cong P \times \mathfrak{g}$, we may write $\omega \in \Omega^1(P, \mathfrak{g})^G$.

Course summary

Today is the last day! Let's summarize what we've learned so far.

1. A principal G-bundle is a fiber bundle $\pi : P \to B$ with a right G-action $P \curvearrowright G$ so that the fibers of P are G-torsors.
2. A principal connection is a G-equivariant splitting
$$0 \longrightarrow VP \underset{\overset{\omega}{\dashleftarrow}}{\longrightarrow} TP \xrightarrow{d\pi} \pi^* TB \longrightarrow 0,$$
and since $VP \cong P \times \mathfrak{g}$, we may write $\omega \in \Omega^1(P, \mathfrak{g})^G$.
3. We have a covariant exterior derivative $d_\omega = d \circ h^{\otimes k}$ acting on $\Omega^k(P, \mathfrak{g})$, where h is the projection $TP \to HP = \ker \omega$. We define $d_\omega \omega = \Omega \in \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$, which vanishes if and only if HP is integrable.

Course summary

Today is the last day! Let's summarize what we've learned so far.

1. A principal G-bundle is a fiber bundle $\pi : P \to B$ with a right G-action $P \curvearrowright G$ so that the fibers of P are G-torsors.
2. A principal connection is a G-equivariant splitting
$$0 \longrightarrow VP \underset{\overset{\omega}{\dashleftarrow}}{\longrightarrow} TP \xrightarrow{d\pi} \pi^* TB \longrightarrow 0,$$
and since $VP \cong P \times \mathfrak{g}$, we may write $\omega \in \Omega^1(P, \mathfrak{g})^G$.
3. We have a covariant exterior derivative $d_\omega = d \circ h^{\otimes k}$ acting on $\Omega^k(P, \mathfrak{g})$, where h is the projection $TP \to HP = \ker \omega$. We define $d_\omega \omega = \Omega \in \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$, which vanishes if and only if HP is integrable.
4. The curvature Ω descends to a form $F_\omega \in \Omega^2(P, \mathfrak{g}_P)$ with $\Omega = \pi^* F_\omega$, where $\mathfrak{g}_P = (P \times \mathfrak{g})/G$ for the adjoint action $G \curvearrowright \mathfrak{g}$. In the case $G = \mathrm{U}(1)$ we have $c_1(P) = [F_\omega] \in H^2(B; \mathbb{R})$.

Induced connections

Let $\overline{\pi} : P \to B$ be a principal G-bundle. Recall that our notion of connection was developed so that we could construct a notion of parallel transport: for any path $\gamma : [0,1] \to B$ and $p \in P$ with $\overline{\pi}(p) = \gamma(0)$, we construct a unique lift $\widetilde{\gamma}_p : [0,1] \to P$ satisfying $\widetilde{\gamma}_p(0) = p$.

Induced connections

Let $\overline{\pi} : P \to B$ be a principal G-bundle. Recall that our notion of connection was developed so that we could construct a notion of parallel transport: for any path $\gamma : [0,1] \to B$ and $p \in P$ with $\overline{\pi}(p) = \gamma(0)$, we construct a unique lift $\widetilde{\gamma}_p : [0,1] \to P$ satisfying $\widetilde{\gamma}_p(0) = p$.

Equipped with such notion of parallel transport, we can induce a notion of parallel transport on any associated bundle to P. Recall that if $G \curvearrowright F$, we have an action $(p, f).g = (p.g, g^{-1}.f)$ of G on $P \times F$, and the quotient $E = (P \times F)/G$ is a bundle $\pi : E \to B$ with fiber F.

Induced connections

Let $\overline{\pi} : P \to B$ be a principal G-bundle. Recall that our notion of connection was developed so that we could construct a notion of parallel transport: for any path $\gamma : [0,1] \to B$ and $p \in P$ with $\overline{\pi}(p) = \gamma(0)$, we construct a unique lift $\widetilde{\gamma}_p : [0,1] \to P$ satisfying $\widetilde{\gamma}_p(0) = p$.

Equipped with such notion of parallel transport, we can induce a notion of parallel transport on any associated bundle to P. Recall that if $G \curvearrowright F$, we have an action $(p,f).g = (p.g, g^{-1}.f)$ of G on $P \times F$, and the quotient $E = (P \times F)/G$ is a bundle $\pi : E \to B$ with fiber F.

The let $\gamma : [0,1] \to B$ be a path, let $e = (p,f) \bmod G \in E$ such that $\pi(e) = \gamma(0)$. We define

$$\widetilde{\gamma}_e(t) = (\widetilde{\gamma}_p(t), f) \bmod G.$$

Induced connections

Let $\overline{\pi} : P \to B$ be a principal G-bundle. Recall that our notion of connection was developed so that we could construct a notion of parallel transport: for any path $\gamma : [0,1] \to B$ and $p \in P$ with $\overline{\pi}(p) = \gamma(0)$, we construct a unique lift $\widetilde{\gamma}_p : [0,1] \to P$ satisfying $\widetilde{\gamma}_p(0) = p$.

Equipped with such notion of parallel transport, we can induce a notion of parallel transport on any associated bundle to P. Recall that if $G \curvearrowright F$, we have an action $(p, f).g = (p.g, g^{-1}.f)$ of G on $P \times F$, and the quotient $E = (P \times F)/G$ is a bundle $\pi : E \to B$ with fiber F.

The let $\gamma : [0,1] \to B$ be a path, let $e = (p, f) \bmod G \in E$ such that $\pi(e) = \gamma(0)$. We define

$$\widetilde{\gamma}_e(t) = (\widetilde{\gamma}_p(t), f) \bmod G.$$

Since principal connections are G-equivariant, we have $\gamma_{p.g}(t) = \gamma_p(t).g$, and hence if we write $e = (p.g, g^{-1}.f) \bmod G$ instead, we get

$$(\widetilde{\gamma}_{p.g}(t), g^{-1}.f) \bmod G = (\widetilde{\gamma}_p(t).g, g^{-1}.f) \bmod G = (\widetilde{\gamma}_p(t), f) \bmod G.$$

Induced connections

Since the horizontal lifting of vectors from TB up to TE is simply the infinitesimal version of parallel transport, we may also express our notion of induced connection as follows.

Induced connections

Since the horizontal lifting of vectors from TB up to TE is simply the infinitesimal version of parallel transport, we may also express our notion of induced connection as follows.

Let $\overline{\pi} : P \to B$ be a principal G-bundle with parallel transport $(\gamma, p) \mapsto \widetilde{\gamma}_p$ given by a connection ω. Then any associated bundle $\pi : E = (P \times F)/G \to B$ has an induced connection given by the splitting

$$0 \longrightarrow VE \longrightarrow TE \underset{h_\omega}{\overset{d\pi}{\rightleftarrows}} \pi^* TB \longrightarrow 0,$$

where h_ω is defined as follows. Every $(e, X) \in \pi^* TB$ is of the form $\frac{\partial}{\partial t}\gamma$ for some path γ in B with $\gamma(0) = \pi(e)$. We define $h_\omega\left(\frac{\partial}{\partial t}\gamma\right) = \frac{\partial}{\partial t}\widetilde{\gamma}_e \in TE$.

Induced connections

Both a right-splitting and a left-splitting of a short exact sequence of vector bundles are equivalent to a direct sum decomposition $TE = VE \oplus HE$ of the middle term. Given such a right-splitting $h_\omega : \pi^* TB \to TE$, consider the associated left-splitting

$$0 \longrightarrow VE \underset{\overset{v_\omega}{\dashleftarrow}}{\longrightarrow} TE \xrightarrow{d\pi} \pi^* TB \longrightarrow 0.$$

Induced connections

Both a right-splitting and a left-splitting of a short exact sequence of vector bundles are equivalent to a direct sum decomposition $TE = VE \oplus HE$ of the middle term. Given such a right-splitting $h_\omega : \pi^* TB \to TE$, consider the associated left-splitting

$$0 \longrightarrow VE \underset{\overset{v_\omega}{\dashleftarrow}}{\longrightarrow} TE \xrightarrow{d\pi} \pi^* TB \longrightarrow 0.$$

When the fiber F of $E = (P \times F)/G$ is a vector space, with $F \curvearrowleft G$ by linear maps, then it is an exercise to see that $VE \cong \pi^* E$.

Induced connections

Both a right-splitting and a left-splitting of a short exact sequence of vector bundles are equivalent to a direct sum decomposition $TE = VE \oplus HE$ of the middle term. Given such a right-splitting $h_\omega : \pi^* TB \to TE$, consider the associated left-splitting

$$0 \longrightarrow VE \underset{\curvearrowleft v_\omega}{\longrightarrow} TE \xrightarrow{d\pi} \pi^* TB \longrightarrow 0.$$

When the fiber F of $E = (P \times F)/G$ is a vector space, with $F \curvearrowleft G$ by linear maps, then it is an exercise to see that $VE \cong \pi^* E$.

Given a section $(s : B \to E) \in \Omega^0(B, E)$, we may define a 1-form $(\nabla s : TB \to E) \in \Omega^1(B, E)$ via

$$\nabla s : TB \xrightarrow{ds} s^* TE \xrightarrow{s^* v_\omega} s^* VE = s^* \pi^* E = E.$$

Induced connections

Both a right-splitting and a left-splitting of a short exact sequence of vector bundles are equivalent to a direct sum decomposition $TE = VE \oplus HE$ of the middle term. Given such a right-splitting $h_\omega : \pi^* TB \to TE$, consider the associated left-splitting

$$0 \longrightarrow VE \overset{v_\omega}{\underset{\dashleftarrow}{\longrightarrow}} TE \xrightarrow{d\pi} \pi^* TB \longrightarrow 0.$$

When the fiber F of $E = (P \times F)/G$ is a vector space, with $F \curvearrowleft G$ by linear maps, then it is an exercise to see that $VE \cong \pi^* E$.

Given a section $(s : B \to E) \in \Omega^0(B, E)$, we may define a 1-form $(\nabla s : TB \to E) \in \Omega^1(B, E)$ via

$$\nabla s : TB \xrightarrow{ds} s^* TE \xrightarrow{s^* v_\omega} s^* VE = s^* \pi^* E = E.$$

In words, for $X = \frac{\partial}{\partial t}\gamma$, the vector $\nabla s(X)$ is the infinitesimal vertical displacement along $s \circ \gamma$.

Covariant derivatives

The map $\nabla = v_\omega \circ d : \Omega^0(B, E) \to \Omega^1(B, E)$ is a covariant derivative.

Definition

A covariant derivative on a vector bundle $\pi : E \to B$ is an $\mathbb{R}$-linear map $\nabla : \Omega^0(B, E) \to \Omega^1(B, E)$ satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(B),\ s \in \Omega^0(M, E).$$

Covariant derivatives

The map $\nabla = v_\omega \circ d : \Omega^0(B, E) \to \Omega^1(B, E)$ is a covariant derivative.

Definition

A covariant derivative on a vector bundle $\pi : E \to B$ is an $\mathbb{R}$-linear map $\nabla : \Omega^0(B, E) \to \Omega^1(B, E)$ satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(B),\ s \in \Omega^0(M, E).$$

We can come full circle and use a connection to define a notion of parallel transport. Given a path $\gamma : [0, 1] \to B$, a section $s : B \to E$ satisfying $\nabla s(\gamma'(t)) = 0$ for each $t \in [0, 1]$ gives a lift $s \circ \gamma$ of γ.

Covariant derivatives

The map $\nabla = v_\omega \circ d : \Omega^0(B, E) \to \Omega^1(B, E)$ is a covariant derivative.

Definition

A covariant derivative on a vector bundle $\pi : E \to B$ is an $\mathbb{R}$-linear map $\nabla : \Omega^0(B, E) \to \Omega^1(B, E)$ satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(B),\ s \in \Omega^0(M, E).$$

We can come full circle and use a connection to define a notion of parallel transport. Given a path $\gamma : [0, 1] \to B$, a section $s : B \to E$ satisfying $\nabla s(\gamma'(t)) = 0$ for each $t \in [0, 1]$ gives a lift $s \circ \gamma$ of γ.

From now on, we will write $\nabla_X s$ instead of $\nabla s(X)$.

Curvature in the language of covariant derivatives

Given a covariant derivative $\nabla : \Omega^0(B, E) \to \Omega^1(B, E)$, we define the curvature tensor $R^\nabla \in \Omega^2(B, \mathrm{End}(E))$ by

$$R^\nabla(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

for vectors $X, Y \in TB$ and sections $s : B \to E$.

Curvature in the language of covariant derivatives

Given a covariant derivative $\nabla : \Omega^0(B, E) \to \Omega^1(B, E)$, we define the curvature tensor $R^\nabla \in \Omega^2(B, \mathrm{End}(E))$ by

$$R^\nabla(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

for vectors $X, Y \in TB$ and sections $s : B \to E$.

We have not strayed from our original definition of curvature. If $E = (P \times V)/G$ where $G \curvearrowright V$ by a representation $\rho : G \to \mathrm{GL}(V)$, then one may check that we have an isomorphism $\Omega^*(B, \mathrm{End}(E)) \cong \Omega^*_{\mathrm{hor}}(B, \mathrm{End}(V))^G$. Then for a connection ω on P and $\nabla = v_\omega \circ d$, we have

$$\dot{\rho} \circ \Omega = R^\nabla \in \Omega^2_{\mathrm{hor}}(B, \mathrm{End}(V))^G,$$

where $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ is the induced map on Lie algebras.

Moduli of curves and the Hodge bundle

(switch to virtual drawing board)

References

- G. Forni, C. Matheus, A. Zorich, *Lyapunov spectrum of invariant subbundles of the Hodge bundle*
- P. Griffiths and J. Harris, *Principles of algebraic geometry*
- S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1
- F. Labourie, *Lectures on Representations of surface groups*
- P. Michor, *Topics in differential geometry*
- T. Walpuski, Notes on the geometry of manifolds, `https://math.mit.edu/~walpuski/18.965/GeometryOfManifolds.pdf`
- My notes on tangent spaces to character varieties on my website