Principal Bundles

Day 4: Chern Classes

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Summary of Day 3

Given a principal G-bundle $\pi : P \twoheadrightarrow B$ with a connection $\omega \in \Omega^1(P, \mathfrak{g})$, the horizontal projection $h : TP \twoheadrightarrow HP = \ker \omega$ gives a covariant exterior derivative $d_\omega \eta = (d\eta) \circ h^{\otimes k+1}$ for $\eta \in \Omega^k(P, \mathfrak{g})$.

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- obstructs $\left(\Omega^*_{\text{hor}}(P, \mathfrak{g})^G, d_\omega\right)$ from being a cochain complex.
- models the electromagnetic field when $B = \mathbb{R}^4$, $G = U(1)$.

The first Chern class

Recall that a connection splits the exact sequence

$$0 \longrightarrow VP \overset{\omega}{\underset{\dashleftarrow}{\longrightarrow}} TP \xrightarrow{d\pi} \pi^* TB \longrightarrow 0,$$

and so we have an isomorphism $d\pi : \ker \omega = HP \xrightarrow{\sim} \pi^* TB$. Let $b \in B$. Then for $p \in \pi^{-1}(b)$ and $X_b \in T_b B$, there is a unique $\widetilde{X}_p \in H_p P$ with $d\pi(\widetilde{X}_p) = X_b$.

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Definition

Let $\pi : P \to B$ be a principal U(1)-bundle with connection ω. Then we define $F_\omega \in \Omega^2(B, \mathfrak{u}(1))$ via

$$F_\omega(X_b, Y_b) = \Omega(\widetilde{X}_p, \widetilde{Y}_p)$$

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Observe that this gives $\Omega = \pi^* F_\omega$.

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$$F(X_b, Y_b) = \Omega(\widetilde{X}_p, \widetilde{Y}_p) \qquad \forall b \in B,\ p \in \pi^{-1}(b)$$

Recall that since our connection is principal, we have $H_{p.g}P = dR_g(H_pP)$, and hence $\widetilde{X}_{p.g} = dR_g(\widetilde{X}_p)$ for every $g \in \mathrm{U}(1)$.

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$$\Omega(\widetilde{X}_{p.g}, \widetilde{Y}_{p.g}) = \Omega(dR_g(\widetilde{X}_p), dR_g(\widetilde{Y}_p)) = \mathrm{Ad}_{g^{-1}} \circ \Omega(\widetilde{X}_p, \widetilde{Y}_p) = \Omega(\widetilde{X}_p, \widetilde{Y}_p),$$

where the final equality follows since U(1) is abelian, and hence $\mathrm{Ad}_{g^{-1}} = \mathrm{Id}_{\mathfrak{u}(1)}$ for every $g \in \mathrm{U}(1)$. We conclude that F_ω is independent of the choice of $p \in \pi^{-1}(b)$.

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Now observe that

$$dF_\omega(X, Y) = d\Omega(\widetilde{X}, \widetilde{Y}) = d\Omega(h(\widetilde{X}), h(\widetilde{Y})) = d_\omega\Omega(\widetilde{X}, \widetilde{Y}) = 0,$$

where the final equality follows from the Bianchi identity.

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We have now shown that $F_\omega(X, Y) = \Omega(\widetilde{X}, \widetilde{Y})$ is a well-defined closed $\mathfrak{u}(1)$-valued 2-form on B. But $\mathfrak{u}(1) = \mathbb{R}$, and so this is just an ordinary closed 2-form. We conclude that we may give the following definition.

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Definition (First Chern class)

Let $\pi : P \to B$ be a principal U(1)-bundle with connection ω. We define the first Chern class of P by

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A homotopy argument shows that $c_1(P)$ does not depend on the choice of ω, hence our notation.

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Theorem

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Proof.

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Proof.

Consider the principal $\mathrm{U}(1)$-bundle $(\pi \times \mathrm{Id}_{\mathbb{R}}) : P \times \mathbb{R} \to B \times \mathbb{R}$ where $P \times \mathbb{R} \curvearrowleft G$ via $(p, t).g = (p.g, t)$.

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$$\omega = (1-t)(\mathrm{proj}_P)^*\omega_0 + t(\mathrm{proj}_P)^*\omega_1.$$

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Thus $F_{\omega_j} = \iota_{B,j}^* F_\omega$. Since $\iota_{B,0} \simeq \iota_{B,1}$, we conclude that $[F_{\omega_0}] = [F_{\omega_1}]$. □

The first Chern class

In complex geometry, one often sees the first Chern class defined via the long exact sequence in cohomology that is induced by the exponential exact sequence.

We will not discuss this definition, as it involves a background with Čech cohomology that we do not assume, but we remark that it differs from our definition by a constant multiple. See page 141 of Griffiths-Harris for the proof of this.

Symplectic manifolds

Definition

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Then $\alpha = -d\theta$ is closed because it is exact, and the coordinate expression $\alpha|_{T^*U} = \sum_j dq_j \wedge dp_j$ shows that α is nondegenerate.

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- Let h be a hermitian metric on a complex manifold M. Then h is Kähler if and only if $\alpha = -\mathrm{Im} h \in \bigwedge^2 T^*M$ is symplectic.

Hamiltonian actions

Let (M, α) be a symplectic manifold, and let $M \curvearrowleft G$ via symplectomorphisms. That is, $R_g^* \alpha = \alpha$ for every $g \in G$. As we've seen before, each $A \in \mathfrak{g}$ defines a vector field $\widehat{A}$ on M via $\widehat{A} = \frac{\partial}{\partial t} R_{\exp(tA)}$.

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We will be concerned with analyzing the level sets $\mu^{-1}(a)$ for Hamiltonian U(1)-actions.

The reduced space

Let (M, α) be a symplectic manifold, and let $M \circlearrowleft \mathrm{U}(1)$ be a Hamiltonian action. We see that μ is $\mathrm{U}(1)$-invariant, i.e. μ is constant along the flowline of any $B \in \mathfrak{u}(1) = \mathbb{R}$, as follows.

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One may rephrase this by saying that each level set $\mu^{-1}(a)$ is preserved by the U(1)-action. Suppose μ is proper and the actions $\mu^{-1}(a) \curvearrowleft \mathrm{U}(1)$ are free. Hence for each $a \in \mathfrak{u}(1)^*$, we have a principal U(1)-bundle

$$\pi_a : \mu^{-1}(a) \to \mu^{-1}(a)/\mathrm{U}(1) = M_a.$$

A product model

One may verify that the reduced space $M_a = \mu^{-1}(a)/\mathrm{U}(1)$ admits a unique symplectic form ν_a satisfying $\pi_a^* \nu_a = \alpha|_{\mu^{-1}(a)}$. We seek a convenient model for these M_a so that we may compute how ν_a depends on a.

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Let us consider the product space $\mu^{-1}(0) \times (-\varepsilon, \varepsilon)$, and let $\omega \in \Omega^1(\mu^{-1}(0))$ be a principal connection for the bundle $\pi_0 : \mu^{-1}(0) \to \mu^{-1}(0)/\mathrm{U}(1)$. Let t denote the $(-\varepsilon, \varepsilon)$-coordinate.

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The form $\widetilde{\alpha} = \alpha|_{\mu^{-1}(0)} + d(t\omega) \in \Omega^2(\mu^{-1}(0) \times (-\varepsilon, \varepsilon))$ is symplectic if ε is small enough.

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Let us consider the action $\mu^{-1}(0) \times (-\varepsilon, \varepsilon) \curvearrowleft \mathrm{U}(1)$ given by $(x, t).g = (x.g, t)$. It is an exercise to verify that this action is Hamiltonian with moment map $J : \mu^{-1}(0) \times (-\varepsilon, \varepsilon) \to \mathfrak{u}(1)^* = \mathbb{R}$ given by $J(x, t) = t$.

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It is a consequence of the coisotropic embedding theorem (see Guillemin, pages 25-26), that there is a neighborhood $M \supset U \supset \mu^{-1}(0)$ and a U(1)-equivariant symplectomorphism

$$(U, \alpha) \xrightarrow{\sim} (\mu^{-1}(0) \times (-\varepsilon, \varepsilon), \widetilde{\alpha})$$

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Variation of reduced spaces

For small enough a, we have now shown that

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In particular, when we identify $(M_a, \nu_a) = (M_0, \widetilde{\nu}_a)$ and take cohomology classes, we find that

$$[\nu_a] = [\nu_0] + ac_1(\mu^{-1}(0)).$$

Variation of volume is a polynomial

Let $n = \dim M_a$, and recall that $\nu_a^{\wedge n}$ is a volume form. The expression $[\nu_a] = [\nu_0] + ac_1(\mu^{-1}(0))$ then gives

$$\mathrm{vol}(M_a) = \sum_{k=0}^{n} \binom{n}{k} a^k \int_{M_0} [\nu_0]^{n-k} \smile c_1(\mu^{-1}(0))^k.$$

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