Principal Bundles

Day 3: Curvature

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Summary of Day 2

Given a principal bundle $\pi : P \to B$, we would like to have a way of uniquely lifting paths from B up to P. We do this by splitting $TP = HP \oplus VP$ so that every tangent vector in TB lifts to a unique vector in HP. Succinctly, ω_p is the projection $T_pP \to V_pP$ with kernel H_pP.

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Definition

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : P \to B$ be a principal G-bundle. A connection 1-form on π is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that

$$\omega\left(\frac{\partial}{\partial t} R_{\exp(tA)}\right) = A \qquad \forall A \in \mathfrak{g}, \text{ (Projection to vertical)}$$

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \qquad \forall g \in G. \text{ (} G\text{-equivariance)}$$

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On Day 2, we referred to the choice of splitting $TP = HP \oplus VP$ as the connection. We will now succinctly refer to ω as the connection.

The covariant exterior derivative

Recall that the horizontal projection $h : TP \twoheadrightarrow HP$ gives a covariant exterior derivative $d_\omega \eta = (d\eta) \circ h^{\otimes k+1}$ for $\eta \in \Omega^k(P, \mathfrak{g})$. Further, recall that $\Omega^*(P, \mathfrak{g})$ is equipped with a bracket $[\eta, \kappa]_\wedge$ defined as an alternating sum of the $\mathfrak{g}$-brackets of η and κ evaluated on permutations of input vectors.

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On Day 2, we showed that for $\eta \in \Omega^k_{\text{hor}}(P, \mathfrak{g})^G$, we have the formulas

$$d_\omega \eta = d\eta + [\omega, \eta]_\wedge$$
$$d_\omega^2 \eta = \left[d\omega + \tfrac{1}{2}[\omega, \omega]_\wedge, \eta\right]_\wedge .$$

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Today, we will define $\Omega = d_\omega \omega \in \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$ and show that $\Omega = d\omega + \frac{1}{2}[\omega, \omega]_\wedge$, so that the latter formula becomes

$$d_\omega^2 \eta = [\Omega, \eta]_\wedge .$$

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Observe that if $X \in V_p P$, then for any $Y \in T_p P$ we have

$$\Omega_p(X, Y) = (d\omega)_p(h(X), h(Y)) = (d\omega)_p(0, h(Y)) = 0.$$

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Since ω is a principal connection, dR_g respects the direct-sum decomposition $TP = HP \oplus VP$, i.e. $R_g^* \circ h^* = h^* \circ R_g^*$. Thus

$$R_g^* \Omega = R_g^*(h^*(d\omega)) = h^*(d(R_g^* \omega)) = h^*(d(\mathrm{Ad}_{g^{-1}} \circ \omega)) = \mathrm{Ad}_{g^{-1}} \circ \Omega.$$

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We conclude $\Omega \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^G$.

Cartan's structure equation

Theorem (Cartan's structure equation)

Let $\pi : P \to B$ *be a principal* G*-bundle with connection* $\omega \in \Omega^1(P, \mathfrak{g})$*.*
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Proof.

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Proof.

From the definition of $[\cdot, \cdot]_\wedge$, one may check that $[\omega, \omega]_\wedge(X, Y) = 2[\omega(X), \omega(Y)]$. Thus, we must show

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

for all vector fields X, Y.

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for all vector fields X, Y. If X, Y are horizontal, then the above is just the definition of Ω. If $X, Y \in V_pP$, there are $A, B \in \mathfrak{g}$ with $X = \widehat{A}_p$, $Y = \widehat{B}_p$. Applying the coordinate-free expression for the exterior derivative, we have

$$d\omega(\widehat{A}, \widehat{B}) = \widehat{A}(\omega(\widehat{B})) - \widehat{B}(\omega(\widehat{A})) - \omega([\widehat{A}, \widehat{B}]).$$

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Since $\omega(\widehat{B})$ and $\omega(\widehat{A})$ are the constant functions $p \mapsto B, A$, respectively, the first two terms vanish.

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$$(d\omega)_p(X, Y) + [\omega_p(X), \omega_p(Y)] = -[\omega(\widehat{A}), \omega(\widehat{B})] + [\omega(\widehat{A}), \omega(\widehat{B})] = 0.$$

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On the other hand, since $X, Y \in V_pP$, we have $\Omega_p(X, Y) = 0$, so the structure equation reduces to $0 = 0$.

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On the other hand, since $X, Y \in V_pP$, we have $\Omega_p(X, Y) = 0$, so the structure equation reduces to $0 = 0$.

We leave the final case $X \in H_pP$, $Y \in V_pP$ as an exercise. See Theorem II.5.2 of Kobayashi and Nomizu's text for the solution. □

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$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]_\wedge$$

Observe that for the principal G-bundle $G \to \{*\}$, the only connection 1-form is the Maurer-Cartan form $\theta(A) = A$ for $A \in \mathfrak{g}$. Since the base manifold is a point, Ω must vanish identically, and so we recover the Maurer-Cartan equation

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As consequences of Cartan's structure equation, we will derive the following two facts:

- $d_\omega \Omega = 0$ (the Bianchi identity),
- $HP = \ker \omega$ is integrable if and only if $\Omega \equiv 0$.

The Bianchi identity

Theorem (Bianchi identity)

Let $\pi : P \to B$ be a principal G-bundle with connection $\omega \in \Omega^1(P, \mathfrak{g})$. Then

$$d_\omega \Omega = 0.$$

Proof.

By Cartan's structure equation, we have

$\square$

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$$d_\omega \Omega = h^* dd\omega + h^* d(\tfrac{1}{2}[\omega, \omega]_\wedge)$$

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where the final equality follows because $h^*\omega = \omega \circ h$ is a composition of two projections with complementary images. □

Flatness

Let M be an n-dimensional manifold. Recall that a rank r subbundle $E \subset TM$ is integrable if for every $x_0 \in M$, there is an open neighborhood $U \ni x_0$ and a coordinate chart $\varphi : (V \subset \mathbb{R}^n) \to U$ so that

$$E_x = d\varphi \left(\mathrm{span} \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_r} \right) \right) \qquad \forall x \in U.$$

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Theorem (Frobenius' integrability theorem)

A subbundle $F \subset TM$ is integrable if and only if $[X, Y]_x \in E_x$ for every $x \in M$ whenever X and Y are vector fields with $X_x, Y_x \in E_x$ for every $x \in M$.

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Theorem

We say that a connection ω is flat if $HP = \ker \omega \subset TP$ is integrable. We have that ω is flat if and only if $\Omega \equiv 0$.

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Thus $[X, Y] \in \ker \omega$ if and only if $\Omega(X, Y) = 0$. By Frobenius' integrability theorem, we are done. □

Moving down to the base manifold

Let $\pi : P \to B$ be a principal G-bundle with a connection ω. On neighborhoods $U \subset B$, we always have $F_U \in \Omega^2(U, \mathfrak{g})$ so that $\Omega|_{\pi^{-1}(U)} = \pi^* F_U$.

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It is an exercise to see that we have an isomorphism $\pi^* : \Omega^*(B, \mathfrak{g}_P) \xrightarrow{\sim} \Omega^*_{\mathrm{hor}}(P, \mathfrak{g})^G$. Therefore, d_ω acts on $\Omega^*(B, \mathfrak{g}_P)$.

Maxwell's equations

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$$\nabla \times E = -\frac{\partial}{\partial t} B \qquad \text{(Faraday's law of induction)}$$

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$$\nabla \times E = -\frac{\partial}{\partial t} B \qquad \text{(Faraday's law of induction)}$$

These are called the homogeneous equations, and they relate $E(t)$ and $B(t)$ to each other. There are also two inhomogenous equations, which relation $E(t)$ and $B(t)$ to each other and to:

- a function $\rho(t) : \mathbb{R}^3 \to \mathbb{R}$ for each time t (electric charge density),
- a time-dependent vector field $J(t)$ on $\mathbb{R}^3$ (electric current).

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\nabla \times E + \frac{\partial}{\partial t} B &= 0 && \text{(Faraday's law of induction)} \\
\nabla \cdot E &= \rho && \text{(Gauss' law)} \\
\nabla \times B - \frac{\partial}{\partial t} E &= J && \text{(Ampère's law with Maxwell's addition)}
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Notice the formal similarity of the first pair and the second pair, especially when ρ and J are 0.

Maxwell's equations

The previous equations are the more classical expressions of Maxwell's laws. We will now stop treating t as an auxiliary parameter and think of it as a coordinate on $\mathbb{R}^4$. In this setting (general relativity), physicists tell us that we ought to endow $\mathbb{R}^4$ with the Lorentzian metric

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$$g(v, w) = -dt(v)dt(w) + \sum_{i=1}^{3} dx_i(v)dx_i(w).$$

This Lorentzian metric induces a natural pairing $\langle \cdot, \cdot \rangle$ on differential forms, and so we may define a Hodge star $\star : \bigwedge_{i=1}^{k} T_p^*\mathbb{R}^4 \to \bigwedge_{i=1}^{4-k} T_p^*\mathbb{R}^4$ via

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mathrm{vol}_g \qquad \alpha, \beta \in \bigwedge_{i=1}^{k} T_p^*\mathbb{R}^4.$$

where vol_g is the volume form determined by the metric g.

Maxwell's equations

Let E_i denote the $\frac{\partial}{\partial x_i}$-component of E, and similarly for B and J. Then define

$$\begin{aligned}
\eta &= E_1 dx_1 + E_2 dx_2 + E_3 dx_3 \in \Omega^1(\mathbb{R}^4), \\
\beta &= B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_3 \in \Omega^2(\mathbb{R}^4), \\
\mathcal{J} &= -\rho dt + J_1 dx_1 + J_2 dx_2 + J_3 dx_3 \in \Omega^1(\mathbb{R}^4).
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$$\mathcal{J} = -\rho dt + J_1 dx_1 + J_2 dx_2 + J_3 dx_3 \in \Omega^1(\mathbb{R}^4).$$

Then for $F = \beta + \eta \wedge dt$ (the electromagnetic field), Maxwell's equations are

$$dF = 0 \qquad \text{(Homogeneous equations)}$$
$$\star d \star F = \mathcal{J} \qquad \text{(Inhomogeneous equations)}$$

Dirac's magnetic monopole

The equation $\nabla \cdot B = 0$ tells us that we cannot have any magnetic monopoles, but we can still model the idea of a magnetic monopole by setting $X = \mathbb{R}^3 \setminus (0,0,0)$ and considering Maxwell's equations on $X \times \mathbb{R}$, where the $\mathbb{R}$ factor represents time.

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Indeed, one may verify that if we set $\eta = 0$ and

$$\beta = \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{3}{2}} (x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_3),$$

then $F = \beta + \eta \wedge dt \in \Omega^2(X \times \mathbb{R})$ satisfies Maxwell's equations for $\mathcal{J} = 0$. Since F is not an exact form, it does not admit an antiderivative on $X \times \mathbb{R}$.

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Let G be one of U(1), $\mathbb{R}$, so that $\mathfrak{g} = \mathbb{R}$. We would like to postulate a principal G-bundle $\pi : P \to X \times \mathbb{R}$ with a connection ω so that $\pi^* F = \Omega = d_\omega \omega$. In this situation, we call ω the electromagnetic potential, and it serves as the next best thing to an antiderivative for F.

Reformulating Maxwell's equations

Let's check that the equation $\pi^*F = \Omega$ makes sense. Recall that we must have $F \in \Omega^2(X \times \mathbb{R}, \mathfrak{g}_P)$ in order for $\pi^*F = \Omega$. Since $\mathfrak{g} = \mathbb{R}$ is abelian, the adjoint action $G \curvearrowright \mathfrak{g}$ is trivial, and so $\mathfrak{g}_P = (X \times \mathbb{R}) \times \mathfrak{g}$. Thus $F \in \Omega^2(X \times \mathbb{R}, \mathfrak{g}) = \Omega^2(X \times \mathbb{R}, \mathfrak{g}_P)$.

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Again because $\mathfrak{g}$ is abelian, we have $d_\omega(\cdot) = d(\cdot) + [\omega, \cdot]_\wedge = d(\cdot)$. Therefore, the conditions $dF = 0$, $\star d \star F = \mathcal{J}$ become $d_\omega F = 0$ (Bianchi), $\star d_\omega \star F = \mathcal{J}$. We have now written Maxwell's equations in the language of principal connections!

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The Yang-Mills equations

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Let G be a Lie group (typically non-abelian) and $\pi : P \to B$ a principal G-bundle with connection ω, where B is equipped with a Hodge star $\star$. Let $F \in \Omega^2(B, \mathfrak{g}_P)$ satisfy $\Omega = \pi^* F$.

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The Yang-Mills equations are then

$$\begin{aligned} d_\omega F &= 0 \qquad \text{(Bianchi)} \\ \star d_\omega \star F &= 0. \end{aligned}$$

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