Principal Bundles

Day 2: Connections

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We work in the category of C^∞ manifolds.

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An example to keep in mind is the frame bundle $FM \twoheadrightarrow M$ of an n-manifold M, whose fiber at $p \in M$ is the collection of bases for the tangent space T_pM. This is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle, where the action comes from changes-of-basis.

Path lifting and parallel transport

Let $\pi : E \twoheadrightarrow B$ be any fiber bundle, and let $\gamma : [0,1] \to B$ be a path in B. Let $x \in \pi^{-1}(\gamma(0))$. In general, there exist many paths $\widetilde{\gamma}_x : [0,1] \to E$ with $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = x$. When $\pi : E \twoheadrightarrow B$ is a covering space, there exists only one such $\widetilde{\gamma}$, but in every other case, there exist infinitely many choices of $\widetilde{\gamma}$.

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Suppose we have a way of assigning a unique $\widetilde{\gamma}_x$ to each (γ, x). We understand the fibers of π as encoding some data attached to point $b \in B$. Then an assignment $(\gamma, x) \mapsto \widetilde{\gamma}_x$ gives a way of transporting an initial piece of data x over $\gamma(0)$ to a piece of data over a different point $\gamma(1)$.

Path lifting and parallel transport

On every fiber of $\pi : E \twoheadrightarrow B$, we have a canonical notion of vertical direction: a tangent vector is vertical if it lies in $\ker(d\pi)$. Note that if $F = \pi^{-1}(b)$ is a fiber, then $T_pF = \ker(d\pi_p)$ for every $p \in F$.

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Suppose $\pi(x) = b$. Every tangent vector $v \in T_bB$ determines a unique coset $d\pi_x^{-1}(v) = \widetilde{v} + \ker(d\pi_x) \subset T_xE$. Suppose we are able to choose a consistent-enough choice of representative $\widetilde{v}$, and let $\gamma : [0,1] \to B$ be a path.

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Then we may lift the velocity vector field $\gamma'(t)$ to a vector field $\widetilde{\gamma'(t)}$ on $\pi^{-1}(\gamma([0,1]))$. We then define $\widetilde{\gamma}_x : [0,1] \to E$ to be the integral curve of this vector field with initial point $x \in \pi^{-1}(b)$.

Definition

Let $\pi : E \twoheadrightarrow B$ be a fiber bundle. Let $VE \twoheadrightarrow E$ be the subbundle of $TE \twoheadrightarrow E$ whose fiber at $x \in E$ is $V_xE = \ker(d\pi_x)$. A connection is a choice of subbundle $HE \twoheadrightarrow E$ of TE whose fiber H_xE at every $x \in E$ satisfies

$$H_xE \oplus V_xE = T_xE.$$

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One may understand this globally as a choice of left-splitting

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Since $HE \cong \pi^* TB$, every $v \in T_x B$ now has a unique lift $\widetilde{v} \in H_x E$, and so we obtain a notion of parallel transport as desired.

Example: Levi-Civita connection

Recall that on a Riemannian manifold M, we have a notion of covariant derivative ∇, and a vector field X over a curve $\gamma : [0,1] \to M$ is parallel if $\nabla_{\gamma'(t)} X = 0$ for every t.

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At every $v \in TM$, define $H_v(TM)$ to be the set of all $X'(0)$, where $X : [0,1] \to TM$ is a parallel vector field over a curve $\gamma : [0,1] \to M$ with $X(0) = v$. Then $H(TM) \twoheadrightarrow TM$ is a connection on the fiber bundle $\pi : TM \twoheadrightarrow M$.

Vector-valued differential forms

Recall that a differential k-form $\omega \in \Omega^k(M)$ on a manifold M is a smooth choice of alternating $\bigotimes_{i=1}^k T_xM \to \mathbb{R}$ for every $x \in M$

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Recall that any $\omega \in \Omega^k(M)$ can be understood as a section of $\bigwedge_{i=1}^k T^*M$. Similarly, if $F \twoheadrightarrow M$ is the trivial vector bundle $M \times V$, any $\omega \in \Omega^k(M, V)$ is a section of $\left(\bigwedge_{i=1}^k T^*M\right) \otimes F$.

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Letting $F \twoheadrightarrow M$ be any vector bundle, one also defines $\Omega^k(M, F)$ as the space of sections of $\left(\bigwedge_{i=1}^k T^*M\right) \otimes F$.

Connection 1-forms

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Conversely, given any VE-valued 1-form ω on E so that ω_p is a projection $T_pE \to V_pE$ for every $p \in E$, we have a connection $\ker \omega = HE \twoheadrightarrow E$.

Vertical bundles: The principal case

Let $\overline{\pi} : P \twoheadrightarrow B$ be a principal G-bundle, and let $\mathfrak{g}$ be the Lie algebra of G. For every $A \in \mathfrak{g}$, we have a vector field $\widehat{A}$ on P given by

$$\widehat{A}_p = \frac{\partial}{\partial t}\bigg|_{t=0} p.\exp(tA) \qquad \forall p \in P.$$

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Let $A_1, A_2, \ldots, A_n$ be any basis for $\mathfrak{g}$. Then $\widehat{A}_1, \widehat{A}_2, \ldots, \widehat{A}_n$ are n linearly independent sections of the rank n vector bundle VP. It follows that $VP \cong P \times \mathfrak{g}$. Therefore $\Omega^1(P, VP) = \Omega^1(P, \mathfrak{g})$.

Vertical bundles: The principal case

Exercise

Show that for $R_g : P \to P$, $R_g(p) = p.g$, we have

$$dR_g(\widehat{A}) = \widehat{\mathrm{Ad}_{g^{-1}}(A)} \qquad \forall A \in \mathfrak{g}.$$

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Solution

Recall that if $\varphi_t : M \to M$ is a flow and X is the vector field $X(p) = \frac{\partial}{\partial t}\big|_{t=0} \varphi_t(p)$, then for any diffeomorphism $\psi : M \to M$, we have $d\psi(X)(p) = \frac{\partial}{\partial t}\big|_{t=0} \psi \circ \varphi_t \circ \psi^{-1}(p)$. Thus

$$\begin{aligned} dR_g(\widehat{A}) &= \frac{\partial}{\partial t}\bigg|_{t=0} R_g \circ R_{\exp(tA)} \circ R_{g^{-1}}(p) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} R_{g^{-1}\exp(tA)g}(p) \\ &= \widehat{\mathrm{Ad}_{g^{-1}}(A)}. \end{aligned}$$

Principal connections

Definition

Let $\overline{\pi} : P \twoheadrightarrow B$ be a principal G-bundle, and let $R_g : P \to P$ be given by $R_g(p) = p.g$. We say a connection $HP \twoheadrightarrow P$ is principal if

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Since ω_p is the projection onto $V_pP \cong \mathfrak{g}$ induced by this decomposition, we have $R_g^*\omega_p(v) = 0$ for horizontal vectors V, and for vertical vectors $\widehat{A}$ we have

$$(R_g^*\omega)_p(\widehat{A}) = \omega_{R_g(p)}(dR_g(\widehat{A})) = \omega_{R_g(p)}(\widehat{\mathrm{Ad}_{g^{-1}}(A)}) = \mathrm{Ad}_{g^{-1}}(A).$$

Thus $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$.

I have tried to include a complete discussion for the sake of being well-motivated, though this leaves us open to the danger of getting lost in the weeds. We will take a short break to digest this material. The following is a summary of what we have done so far.

Definition

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\overline{\pi} : P \twoheadrightarrow B$ be a principal G-bundle. A connection 1-form on $\overline{\pi}$ is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that

$$\omega\left(\frac{\partial}{\partial t} R_{\exp(tA)}\right) = A \qquad \forall A \in \mathfrak{g}, \text{ (Projection to vertical)}$$

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \qquad \forall g \in G. \text{ (G-equivariance)}$$

We call the subbundle $HP = \ker \omega \subset TP$ the horizontal subbundle of TP.

The covariant exterior derivative

Just as ω is the vertical projection $TP \to VP$ with kernel HP, let us consider the horizontal projection $h : TP \to HP$ with kernel VP. Let us define $h^* : \Omega^k(P, \mathfrak{g}) \to \Omega^k(P, \mathfrak{g})$ by

$$h^*_\omega \eta(X_1, X_2, \ldots, X_k) = \eta(h_\omega(X_1), h_\omega(X_2), \ldots, h_\omega(X_k)) \qquad \eta \in \Omega^k(P, \mathfrak{g}).$$

That is to say, $h^*\eta = \eta \circ h^{\otimes k}$.

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Definition

Let $\overline{\pi} : P \twoheadrightarrow B$ be a principal G-bundle with connection ω. The covariant exterior derivative $d_\omega : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g})$ is given by

$$d_\omega \eta = h^*(d\eta).$$

The covariant exterior derivative

Recall that the wedge product on $\Omega^*(P, \mathbb{R})$ is defined by

$$\psi \wedge \vartheta(X_1, \dots, X_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \text{sign}(\sigma)\psi(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \vartheta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}),$$

where $\psi \in \Omega^k(P, \mathbb{R})$, $\vartheta \in \Omega^\ell(P, \mathbb{R})$.

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where $\psi \in \Omega^k(P, \mathbb{R})$, $\vartheta \in \Omega^\ell(P, \mathbb{R})$.

The Lie algebra $\mathfrak{g}$ does not have a multiplication $\cdot$ as above, but it does have a bracket $[\cdot, \cdot]$, and so we analogously define

$$[\eta, \kappa]_\wedge(X_1, \dots, X_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sign}(\sigma) [\eta(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \kappa(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})],$$

where $\eta \in \Omega^k(P, \mathfrak{g})$, $\kappa \in \Omega^\ell(P, \mathfrak{g})$.

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Theorem

For $\eta \in \Omega^k_{hor}(P, \mathfrak{g})^G$*, we have* $d_\omega \eta = d\eta + [\omega, \eta]_\wedge$.

Proof.

If $(X_1, \ldots, X_{k+1})$ are horizontal vector fields, then $h(X_i) = X_i$ and $\omega(X_i) = 0$.

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$$[\omega, \eta]_\wedge(X_1, \ldots, X_{k+1}) = \frac{1}{k!} \sum_\sigma \omega(X_{\sigma(i)}) \cdot \eta(X_{\sigma(2)}, \ldots, X_{\sigma(k+1)}) = 0.$$

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Therefore the desired identity holds on horizontal vector fields. (Cont'd) □

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Proof.

For a vector field X, let $i_X : \Omega^*(P, \mathfrak{g}) \to \Omega^{*-1}(P, \mathfrak{g})$ be the insertion operator, and recall Cartan's magic formula $\mathcal{L}_X = i_X d + d i_X$.

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$$i_{\widehat{A}}(d\eta + [\omega, \eta]_\wedge) = \mathcal{L}_{\widehat{A}} \eta + [i_{\widehat{A}} \omega, \eta]_\wedge - [\omega, i_{\widehat{A}} \eta]_\wedge$$

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\begin{aligned}
i_{\widehat{A}}(d\eta + [\omega, \eta]_\wedge) &= \mathcal{L}_{\widehat{A}} \eta + [i_{\widehat{A}} \omega, \eta]_\wedge - [\omega, i_{\widehat{A}} \eta]_\wedge \\
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Proof.

For a vector field X, let $i_X : \Omega^*(P, \mathfrak{g}) \to \Omega^{*-1}(P, \mathfrak{g})$ be the insertion operator, and recall Cartan's magic formula $\mathcal{L}_X = i_X d + di_X$. For $A \in \mathfrak{g}$, notice that $\mathcal{L}_{\widehat{A}} \eta = \frac{\partial}{\partial t}\big|_{t=0} R^*_{\exp(tA)} \eta = \frac{\partial}{\partial t}\big|_{t=0} \mathrm{Ad}_{\exp(-tA)} \circ \eta = -[A, \eta]$. Since $i_{\widehat{A}} \eta = 0$, we have $i_{\widehat{A}} d\eta = i_{\widehat{A}} d\eta + di_{\widehat{A}} \eta = \mathcal{L}_{\widehat{A}} \eta$. Then we have

$$\begin{aligned} i_{\widehat{A}}(d\eta + [\omega, \eta]_\wedge) &= \mathcal{L}_{\widehat{A}} \eta + [i_{\widehat{A}} \omega, \eta]_\wedge - [\omega, i_{\widehat{A}} \eta]_\wedge \\ &= -[A, \eta] + [A, \eta] \end{aligned}$$

□

The covariant exterior derivative

Theorem

For $\eta \in \Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^G$, we have $d_\omega \eta = d\eta + [\omega, \eta]_\wedge$.

Proof.

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Finally, $i_{\widehat{A}} d_\omega \eta = i_{h(\widehat{A})} d\eta = 0$. We conclude that the desired equation holds on any combination of horizontal and vertical vectors, and hence holds identically, as desired. □

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Therefore

$$\cdots \xrightarrow{d_\omega} \Omega^{k-1}_{\text{hor}}(P,\mathfrak{g})^G \xrightarrow{d_\omega} \Omega^k_{\text{hor}}(P,\mathfrak{g})^G \xrightarrow{d_\omega} \Omega^{k+1}_{\text{hor}}(P,\mathfrak{g})^G \xrightarrow{d_\omega} \cdots$$

is a cochain complex if $d\omega + \frac{1}{2}[\omega,\omega]_\wedge$ vanishes. Tomorrow we will see that $d\omega + \frac{1}{2}[\omega,\omega]_\wedge$ is equal to the curvature $\Omega = d_\omega\omega$.

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