Principal Bundles

Day 1: Introduction

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June 15, 2020

Fiber bundles

Throughout this minicourse, we will work in the category of C^∞ manifolds.

Definition (Fiber Bundle)

A surjection $\pi : E \twoheadrightarrow B$ is a *fiber bundle with fiber* F if, for every $b \in B$, there is an open neighborhood $U \ni b$ so that we have a diffeomorphism

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If $B = \bigcup_\alpha U_\alpha$ is an open cover so that there are diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$, we have functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diffeo}(F)$ given by

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(b, f) = (b, \varphi_{\alpha\beta}(b)(f)) \quad \text{for } b \in U_\alpha \cap U_\beta.$$

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$$B = \bigcup_{\alpha} U_\alpha, \quad \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diffeo}(F)$$

We can construct our original bundle from this data. We have $E \cong \coprod_\alpha (U_\alpha \times F)/\sim$, where $(b \in U_\alpha, f) \sim (b \in U_\beta, \varphi_{\alpha\beta}(b)(f))$ for every $b \in U_\alpha \cap U_\beta$. Then $\pi : E \twoheadrightarrow B$ is induced by the first coordinate projection.

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It is often the case that the $\varphi_{\alpha\beta}$ all have image in some subgroup $G \subset \mathrm{Diffeo}(F)$. Of course G acts on F, but it also acts on itself!

We can construct $P = \coprod_\alpha (U_\alpha \times G)/\sim$, where $(b \in U_\alpha, g) \sim (b \in U_\beta, \varphi_{\alpha\beta}(b)g)$ for every $b \in U_\alpha \cap U_\beta$. We have a bundle $\overline{\pi} : P \twoheadrightarrow B$ induced by the first coordinate projection.

Principal bundles

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A fiber bundle $\overline{\pi} : P \twoheadrightarrow B$ with fiber G is a *principal* G*-bundle* if there is a right action $P \curvearrowleft G$ that preserves the fibers of $\overline{\pi}$, and acts freely and transitively on each fiber.

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Every principal G-bundle can be constructed by such open-covering data.

Example: The tangent and frame bundles

Let M be C^∞ manifold. Then the tangent bundle $\pi : TM \twoheadrightarrow M$ is a fiber bundle with fiber $\mathbb{R}^n$. Given any open cover $M = \bigcup_\alpha U_\alpha$ with diffeomorphisms $\varphi : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, the functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diffeo}(\mathbb{R}^n)$ all have images contained in $\mathrm{GL}_n(\mathbb{R})$.

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We may then construct the *frame bundle* $FM = \coprod_\alpha (U_\alpha \times \mathrm{GL}_n(\mathbb{R}))/\sim$ with $(b \in U_\alpha, A) \sim (b \in U_\beta, \varphi_{\alpha\beta}(b)A)$. Again, the first coordinate projection induces $\overline{\pi} : FM \twoheadrightarrow M$.

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If we interpret the matrix A above as a list of linearly independent vectors $A = (v_1 \mid v_2 \mid \cdots \mid v_n)$ on which the matrix $\varphi_{\alpha\beta}(b)$ acts by change-of-coordinates $(\varphi_{\alpha\beta}(b)v_1 \mid \varphi_{\alpha\beta}(b)v_2 \mid \cdots \mid \varphi_{\alpha\beta}(b)v_n)$, we obtain

$$\overline{\pi}^{-1}(b) = \{(v_1, v_2, \ldots, v_n) \in (T_bM)^n \mid (v_1, v_2, \ldots, v_n) \text{ form a basis of } T_bM\}.$$

Example: Regular coverings

Let $\overline{\pi} : X \twoheadrightarrow Y$ be a regular (i.e. normal, i.e. Galois) covering space with deck group Δ. Then $\overline{\pi} : X \twoheadrightarrow Y$ is a principal Δ-bundle:

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Nobody said we had to make natural constructions! The product space $Y \times \pi_1(Y)$ with right action $(y, \gamma).\gamma' = (y, \gamma\gamma')$ is also a principal $\pi_1(Y)$-bundle.

Associated bundles

We obtained principal bundles from more general fiber bundles, and we can go the other way as well.

Definition (Associated bundle)

Let $\overline{\pi} : P \twoheadrightarrow B$ be a principal G-bundle and let F be a space with a left G-action $G \curvearrowright F$. Then $(P \times F) \curvearrowleft G$ via $(p, f).g = (p.g, g^{-1}.f)$. Then the first coordinate projection induces an *associated G-bundle* $\pi : (P \times F)/G \twoheadrightarrow B$ with fiber F.

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It is somewhat easier to see what is going on locally. For $U \times G \twoheadrightarrow U$, our action is $(U \times G \times F) \curvearrowleft G$ via $(b, h, f).g = (b, hg, g^{-1}.f)$. Every G-orbit of (b, h, f) has a unique representative of the form $(b, 1_G, f')$, namely $(b, h, f).h^{-1}$. Thus the middle factor is superfluous, so we have $(U \times G \times F)/G \cong U \times F$. Hence $(P \times F)/G$ has fiber F.

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Consider the actions $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^1$ via $1.e^{i\theta} = e^{-i\theta}$ and $\mathbb{Z}/2\mathbb{Z} \curvearrowright (0,1)$ via $1.x = 1 - x$. Then:

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- $(E_{\text{nontriv}} \times (0,1))/(\mathbb{Z}/2\mathbb{Z})$ is a Möbius strip

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- Indeed, if a bundle $\pi : E \twoheadrightarrow B$ is isomorphic to a Cartesian product $E \cong B \times F$, then the diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ can all be chosen so that the functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Diffeo}(F)$ all have images in $\{1\} \subset \text{Diffeo}(F)$.

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- For the tangent bundle $TM \twoheadrightarrow M$, we said the functions $\varphi_{\alpha\beta} \to \mathrm{Diffeo}(\mathbb{R}^n)$ all have images in $\mathrm{GL}_n(\mathbb{R})$, but it we choose a Riemannian metric on M and compatible diffeomorphisms φ_α, then the functions $\varphi_{\alpha\beta}$ will all have images in $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$.

Definition (Reduction of structure group)

Let G be a group and $H \leq G$ a subgroup. If we have a principal G-bundle $P \twoheadrightarrow B$ and a principal H-bundle $Q \twoheadrightarrow B$, then a fiber-preserving embedding $\psi : Q \hookrightarrow P$ is a *reduction of structure group* if ψ is H-equivariant:

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A principal G-bundle $P \twoheadrightarrow B$ admits a reduction of structure group $\psi : Q \hookrightarrow P$ if and only if B admits an open covering $B = \bigcup_\alpha U_\alpha$ so that there are functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ such that $P \cong \coprod_\alpha (U_\alpha \times G)/\sim$, where $(b \in U_\alpha, g) \sim (b \in U_\beta, \varphi_{\alpha\beta}(b)g)$ for every $b \in U_\alpha \cap U_\beta$

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We have a reduction of structure group $\psi : Q \hookrightarrow P$ *if and only if the functions* $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ *can be taken with image lying in* H*.*

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Proof sketch.

If $\varphi_{\alpha\beta}$ has image in H, then $Q = \coprod_\alpha (U_\alpha \times H)/((b,h) \sim (b, \varphi_{\alpha\beta}(b)h))$ is a principal H-bundle, and ψ can be defined on every $U_\alpha \times H$ by $\psi|_{U_\alpha \times H}(b,h) = (b,h) \in U_\alpha \times G$.

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If we start with a reduction $\psi : Q \hookrightarrow P$, then P is isomorphic to the associated H-bundle $(Q \times G)/H$ via the map

$$\begin{aligned} \Psi : (Q \times G)/H &\to P \\ (q,g) \bmod H &\mapsto \psi(q).g \end{aligned}$$

It is left as an exercise to check that Ψ is a well-defined isomorphism. □

Special cases of reduction of structure group

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A reduction to the trivial subgroup $\{1\} \subset G$ is a fiber-preserving $\{1\}$-equivariant map $\psi : B \cong B \times \{1\} \hookrightarrow P$. Equivariance here is vacuous, and fiber-preserving reduces to the condition $\overline{\pi} \circ \psi = \mathrm{Id}_B$. Thus a reduction to $\{1\}$ is the same thing as a section of $\overline{\pi} : P \twoheadrightarrow B$. We conclude that P is a trivial bundle $P \cong B \times G$ if and only if $\overline{\pi}$ has a section.

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A reduction to $G = G$ is a fiber-preserving G-equivariant diffeomorphism $\psi : P \xrightarrow{\sim} P$. We call such maps *gauge transformations* of P. A gauge transformation induces an automorphism of any associated bundle $(P \times F)/G$ via

$$(p, f)_{/G} \mapsto (\psi(p), f)_{/G}$$

Example: Tensorial structures reduce the frame bundle

Let $\overline{\pi} : FM \twoheadrightarrow M$ be the frame bundle of M. Recall that FM is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle whose fiber at $b \in M$ is the set of bases of T_bM.

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If M admits a volume form, let $SM \twoheadrightarrow M$ be the principal $\mathrm{SL}_n(\mathbb{R})$-bundle whose fiber at $b \in M$ is the set of bases of T_bM with volume 1. The inclusion $SM \hookrightarrow FM$ is a reduction of structure group to $\mathrm{SL}_n(\mathbb{R})$.

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If M admits a Riemannian metric, let $OM \twoheadrightarrow M$ be the principal $\mathrm{O}(n)$-bundle whose fiber at $b \in M$ is the set of orthonormal bases of T_bM. The inclusion $OM \hookrightarrow FM$ is a reduction of structure group to $\mathrm{O}(n)$.

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When n is even, an *almost-complex* structure is a bundle isomorphism $J : TM \xrightarrow{\sim} TM$ with $(J|_{T_bM})^2 = -\mathrm{Id}_{T_bM}$ for every $b \in M$. Let $CM \twoheadrightarrow M$ be the principal $\mathrm{GL}_{n/2}(\mathbb{C})$-bundle whose fiber at $b \in M$ is the set of bases $\{v_i\}_{i=1}^n$ of T_bM with $Jv_{2k} = v_{2k+1}$ for every k. The inclusion $CM \hookrightarrow M$ is a reduction of structure group to $\mathrm{GL}_{n/2}(\mathbb{C})$.

Another perspective on Riemannian metrics

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Standard proof.

By the Whitney embedding theorem, there exists some $N \in \mathbb{N}$ so that there is a smooth embedding $M \hookrightarrow \mathbb{R}^N$. Let g_{Euc} denote the standard Euclidean metric on $\mathbb{R}^N$. Setting $g := g_{\text{Euc}}|_M$, we are done. □

Another perspective on Riemannian metrics

Theorem

Every C^∞ manifold M admits a Riemannian metric g.

Standard proof.

By the Whitney embedding theorem, there exists some $N \in \mathbb{N}$ so that there is a smooth embedding $M \hookrightarrow \mathbb{R}^N$. Let g_{Euc} denote the standard Euclidean metric on $\mathbb{R}^N$. Setting $g := g_{\text{Euc}}|_M$, we are done. □

This is a nice proof, but it would be satisfying to know if there was also a nice proof that does not rely on the Whitney embedding theorem.

Lemma

If F is contractible, then every fiber bundle $\pi : E \to B$ with fiber F has a section.

Proof.

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Let B have a CW-structure such that every k-cell lies in a neighborhood $U \subset B$ over which the bundle can be trivialized $\pi^{-1}(U) \cong U \times F$. Then a section over a k-cell c_k is equivalent to a map $c_k \to F$. Let $B^{(k)}$ denote the k-skeleton of B. We define a section $\sigma : B \to E$ by induction on k.

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For each $b \in B^{(0)}$, we may pick any point in the fiber over b, thereby defining $\sigma|_{B^{(0)}} : B^{(0)} \to E$. Now suppose we have already defined $\sigma|_{B^{(k-1)}}$ for $k > 0$, and let c_k be any k-cell of B. We already have $\partial c_k (\cong S^{k-1}) \to F$, and since $\pi_k(F) = 0$, this extends to a map $c_k \to F$. These maps $c_k \to F$ define $\sigma|_{B^{(k)}}$. By induction, we have a section defined on all of B. □

Another perspective on Riemannian metrics

Let $\mathrm{P}_n(\mathbb{R})$ denote the space of symmetric positive-definite $n \times n$ matrices over $\mathbb{R}$, and let $\pi : TM \to M$ be the tangent bundle. Given a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ over an open subset $U_\alpha \subset M$, a Riemannian metric over U_α is just a choice of section $g_\alpha : U_\alpha \to U_\alpha \times \mathrm{P}_n(\mathbb{R})$.

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Let $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathsf{GL}_n(\mathbb{R})$ be a transition function for TM. A Riemannian metric over $U_\alpha \cup U_\beta$ is a pair of g_α, g_β with $g_\beta(p) = \varphi_{\alpha\beta}(p) g_\alpha(p) \varphi_{\alpha\beta}(p)^\top$. Therefore a Riemannian metric on M is a section of the associated $\mathsf{GL}_n(\mathbb{R})$-bundle $E = \coprod_\alpha (U_\alpha \times \mathrm{P}_n(\mathbb{R}))/\sim$ with fiber $\mathsf{P}_n(\mathbb{R})$.

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By the polar decomposition for real matrices, we have $\mathsf{P}_n(\mathbb{R}) \cong \mathsf{GL}_n(\mathbb{R})/\mathsf{O}(n)$. By the Gram-Schmidt procedure, $\mathsf{GL}_n(\mathbb{R})/\mathsf{O}(n)$ is contractible. Therefore, by the previous lemma, E has a section. That is, there exists a Riemannian metric on M.

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