Lower Bound Algebras and Stanley-Reisner Complexes

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REU Project with Greg Muller and Jenna Rajchgot

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Outline

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1. Lower Bound Algebras from Quivers
2. Cycle Relations
3. Lower Bound Ideals
4. Stanley-Reisner Complexes

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To each vertex i of a quiver Q, we associate

$$x_i' = x_i^{-1}\left(\prod_{j\leftarrow i} x_j^{\mu_{ij}} + \prod_{j\rightarrow i} x_j^{\mu_{ji}}\right) \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$$

Definition (Lower Bound Algebra)

The lower bound algebra $\mathcal{L}$ associated to a quiver Q is the subring $\mathbb{C}[x_1, \ldots, x_n, x_1', \ldots, x_n'] \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Given an arbitrary quiver Q, what relations hold in $\mathcal{L}$ other than the above definitional relations?

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Cycle Relations

Theorem (Cycle Relations)

Whenever a quiver Q contains a cycle $j_1 \to j_2 \to \cdots \to j_k \to j_1$, the product $x'_{j_1} \cdots x'_{j_k}$ can be expressed as a polynomial f in the variables x_i and x'_i whose terms contain fewer than k of the x'_i. Thus $x'_{j_1} \cdots x'_{j_k} - f = 0$.

Proof Strategy

Suppose Q is the cycle $1 \to 2 \to \cdots \to k \to 1$. We expand the product

$$x'_1 \cdots x'_k = \prod_{i=1}^{k} x_i^{-1} \left(x_{i-1} + x_{i+1} \right).$$

We can characterize the terms of the expansion by the number of factors they have of the form $(x_i^{-1} x_{i+1})(x_{i+1}^{-1} x_i) = 1$.

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Lower Bound Ideals

We define the surjective ring homomorphism

$$\varphi : R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathcal{L},$$

$$x_i \mapsto x_i, \quad y_i \mapsto x_i',$$

and let $K = \ker \varphi$. We call K the lower bound ideal. Now $\mathcal{L} \cong R/K$. Can we find a set of generators for K?

Lower Bound Ideals

We can obtain an ordering $<$ of the monomials of R by specifying an ordering on the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. In our case, we use the ordering induced by

$$y_1 > \cdots > y_n > x_1 > \cdots > x_n. \qquad \text{(Lexicographical Ordering)}$$

- Given a polynomial f, the initial term $\text{in}_< f$ is the term of f that is maximal with respect to $<$.
- Given an ideal I, the initial ideal $\text{in}_< I$ is the ideal generated by the initial terms of polynomials in I.
- This is an example of a monomial ideal: an ideal whose generators are monomials.
- A Gröbner basis for an ideal I is a finite set G of generators of I such that $\{\text{in}_< f \mid f \in G\}$ is a set of generators for $\text{in}_< I$.

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Lower Bound Ideals

The definitional relations give us the polynomials

$$D_i = y_i x_i - \left(\prod_{j \leftarrow i} x_j^{\mu_{ij}} + \prod_{j \rightarrow i} x_j^{\mu_{ji}} \right) \in K$$

and the cycle relations give us the polynomials

$$y_{j_1} \cdots y_{j_k} - f \in K.$$

Theorem (Gröbner Basis for K)

The polynomials D_i and $y_{j_1} \cdots y_{j_k} - f$ form a Gröbner basis for K.

We now have generators for K and for $\text{in}_< K$.

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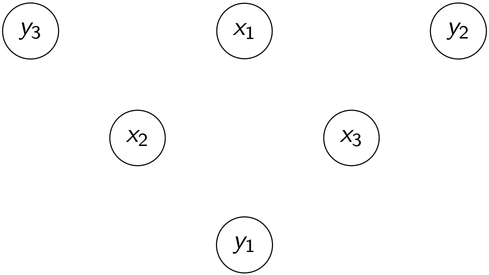
Stanley-Reisner Complexes

Definition (Stanley-Reisner Complex)

Given a monomial ideal I of $k[x_1, \dots, x_n]$, the Stanley-Reisner complex of I is the simplicial complex Δ on $\{x_1, \dots, x_n\}$ such that
$A \in \Delta \iff \prod_{x \in A} x \notin I$.



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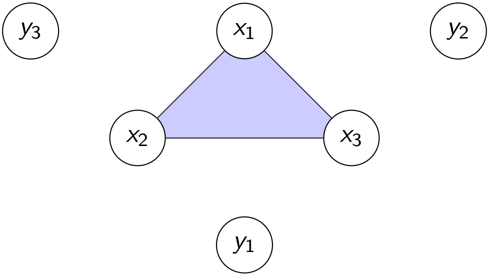

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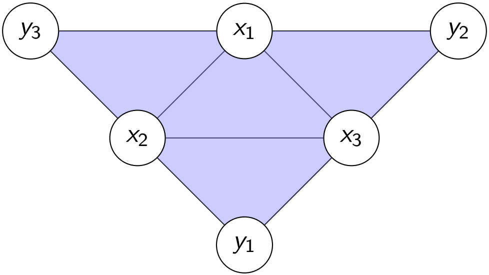

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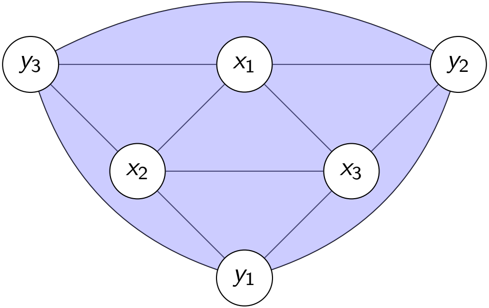
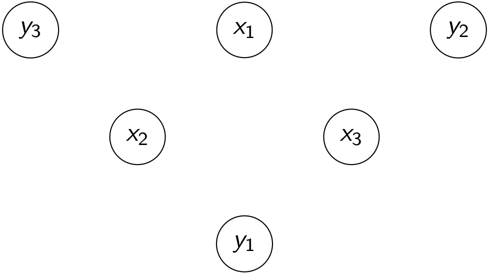


Stanley-Reisner Complexes

Theorem (Topology of some Stanley-Reisner Complexes)

The Stanley-Reisner complex associated to the the initial ideal of a lower bound ideal K is always homeomorphic to an $(n-1)$-ball, except when $y_1 \cdots y_n \notin I$, in which case it is homeomorphic to an $(n-1)$-sphere.



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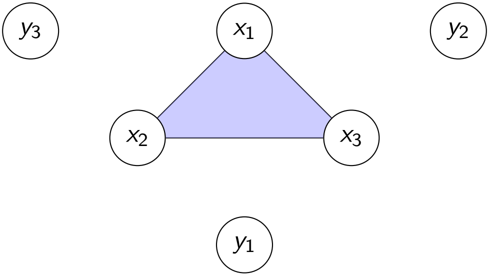

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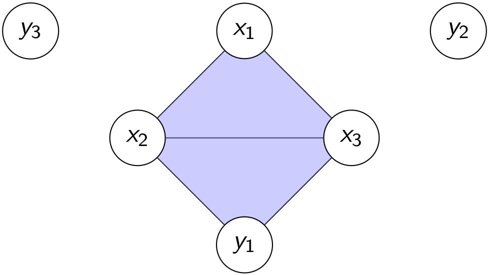

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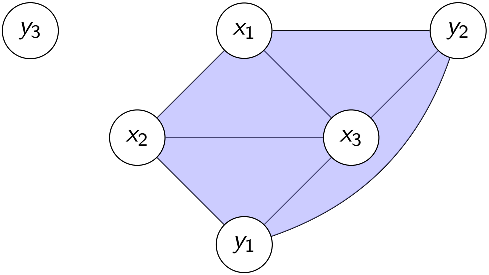

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