Notes on Classical Teichmüller Theory

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1 Beltrami Differentials

One of the key principles of the classical approach to Teichmüller theory is that you can measure how much an orientation-preserving homeomorphism of Riemann surfaces differs from being a biholomorphism at a point by looking at how much it deforms circles in the tangent space at that point. Therefore, we should be able to say how different two complex structures on the same surface are by taking this measurement in the case of the identity map.

We will see that a Beltrami differential is precisely this sort of "measurement." Then the latter claim says that we should guess that the moduli space of complex structures that can be put on a Riemann surface X is the space of Beltrami differentials on X, or perhaps some quotient thereof.

We first analyze the situation in a single tangent space. Given a map $f : X \to Y$ of Riemann surfaces, f is holomorphic at a point $p \in X$ if $Df_p : (T_{\mathbb{R}}X)_p \to (T_{\mathbb{R}}Y)_{f(p)}$ exists and is equivariant with respect to the almost-complex structures J_X and J_Y. That is:

$$Df_p \circ J_X = J_Y \circ Df_p.$$

Remark 1.1. Throughout this section, all almost-complex vector spaces V are assumed to be 2-dimensional over $\mathbb{R}$, with orientation induced by the almost complex structure J_V. △

Consider the following simple fact.

Lemma 1.2. *Let $A : V \to W$ be a linear map of almost-complex vector spaces (V, J_V), (W, J_W). Then there exist unique linear maps $A' : V \to W$ and $A'' : V \to W$, where A' is equivariant with respect to the almost-complex structures, and A'' is anti-equivariant (i.e. $A'' \circ J_V = -J_W \circ A''$), such that*

$$A = A' + A''.$$

Proof. Existence follows from setting

$$A' = \frac{1}{2}\left(A - J_W \circ A \circ J_V\right), \quad A'' = \frac{1}{2}\left(A + J_W \circ A \circ J_V\right).$$

Uniqueness follows from solving for A' and A'' in the equations

$$A \circ J_V = J_W \circ A' - J_W \circ A'', \quad J_W \circ A = J_W \circ A' + J_W \circ A''.$$

□

In terms of the notation from Lemma 1.2, $f : X \to Y$ is holomorphic at p if and only if $Df_p'' = 0$. We adopt the notation

$$\partial f_p \coloneqq Df_p', \quad \overline{\partial} f_p \coloneqq Df_p'',$$

so that

$$f : X \to Y \text{ is holomorphic at p} \iff \overline{\partial} f_p = 0.$$

It should be clear that if you write the equation $\overline{\partial} f_p = 0$ down in coordinates, then you get the equation $\frac{\partial f}{\partial \bar{z}} = 0$.

1.1 The case of a single vector space

Let us consider the setting of almost-complex vector spaces for a bit. A linear map $A : V \to W$ of almost-complex vector spaces (V, J_V), (W, J_W) is equivariant with respect to the almost-complex structures if and only if $A'' = 0$. When $A'' \neq 0$, can we write down some measurement that tells us *how much* the map A fails to be equivariant? The first obvious candidate for such a measurement is the number $\|A''\|$ (this involves choosing a norm $\|\cdot\|$). If we insist that the "measurement" be more than just a number (we like data that can be pulled-back, etc.), then the next obvious candidate is A'' itself. If we further insist that we be able to compare the "non-equivariance" of maps $A : V \to W$ and $B : V \to U$ with different codomains, then we must think a little more.

It is also reasonable to restrict the scope of our discussion slightly: instead of considering arbitrary maps $A : V \to W$, we only consider those that are orientation-preserving isomorphisms. This is a fairly reasonable restriction if our goal is to measure how different a map is from being equivariant with respect to the almost-complex structures; maps that are not orientation-preserving are *very* non-equivariant! If A is orientation-preserving, then both A and A' are necessarily invertible. Furthermore, since $(A')^{-1}$ is equivariant, it is also reasonable to say that composing any map B with $(A')^{-1}$ gives us a map that fails to be equivariant no more and no less than B. With this in mind, it should be reasonable to say that instead of measuring the non-equivariance of $A : V \to W$ with the map $A'' : V \to W$, it's just as good to compose with $(A')^{-1}$, so that we measure non-equivariance with an endomorphism $(A')^{-1} \circ A''$ of V. This solves the problem of not being able to compare the non-equivariance of $A : V \to W$ and $B : V \to U$, since now $(A')^{-1} \circ A''$ and $(B')^{-1} \circ B''$ both lie in $\mathrm{End}_{\mathbb{R}}(V)$. We now have what seems to be a rather acceptable measurement.

Definition 1.3 (Beltrami forms)**.** Given an orientation-preserving map $A : V \to W$ of almost-complex vector spaces, the *Beltrami form* of A is the map

$$\mu(A) := (A')^{-1} \circ A'' \in \mathrm{End}_{\mathbb{R}}(V).$$

Note that this map anti-commutes with J_V. Using the notation of Lemma 1.4 below, that lemma shows that $\|\mu(A)\| = \frac{|b|}{|a|} < 1$. Therefore we define the *space of all Beltrami forms on* V as follows.

$$\mathrm{Bel}(V) := \{\mu \in \mathrm{End}_{\mathbb{R}}(V) \mid J_V \circ \mu = -\mu \circ J_V,\ \|\mu\| < 1\}.$$

Under the usual complex manifold structure on

$$\{\mu \in \mathrm{End}_{\mathbb{R}}(V) \mid J_V \circ \mu = -\mu \circ J_V\} \cong \{\mu \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}) \mid \exists c \in \mathbb{C},\ \mu(z) = c\bar{z}\} \cong \mathbb{C},$$

the space $\mathrm{Bel}(V)$ is biholomorphic to the unit disk $\mathbb{D}$. △

Lemma 1.4 (Exercise 4.8.5 of [Hub06])**.** *Let $A : V \to W$ be a linear map of almost-complex vector spaces. By choosing bases for V and W, we may suppose* *without loss of generality that* $(V, J_V) = (W, J_W) = (\mathbb{C}, i)$. *Then there are* $a, b \in \mathbb{C}$ *with* $A'(z) = az$ *and* $A''(z) = b\bar{z}$. *Then we have*

$$\frac{\|A\|^2}{\det A} = \frac{|a| + |b|}{|a| - |b|}. \tag{1}$$

It clearly follows from (1) *that*

(i) *A is invertible if and only if* $|a| \neq |b|$,

(ii) *A preserves orientation if* $|a| > |b|$ *and reverses orientation if* $|a| < |b|$.

Proof. We can compute the operator norm using any inner product, so let us endow $\mathbb{C}$ with the standard inner product on $\mathbb{R}^2$. By the singular-value decomposition, we may write $A = SDT$, where $S, T \in SO(2)$ and D is diagonal. Let $|\lambda_1| > |\lambda_2|$ be the eigenvalues of D. Then

$$\frac{\|A\|^2}{\det A} = \frac{\lambda_1^2}{\lambda_1 \lambda_2} = \frac{\lambda_1}{\lambda_2}.$$

Therefore we will be done if we can show that $\lambda_1 = |a| + |b|$ and $\lambda_2 = |a| - |b|$. Note that $|\lambda_1| = \max\{\|Av\| \mid v \in V\}$ while $|\lambda_2| = \min\{\|Av\| \mid v \in V\}$.

It is easy to see that $\|Av\|$ is maximized when v makes an angle of $\frac{1}{2}(\arg b - \arg a)$ with the real axis, and A rotates such v by $\arg a$ and then scales the result by $|a| + |b|$. It is also clear that $\|Av\|$ is minimized when v makes an angle of $\frac{1}{2}(\arg b - \arg a) + \frac{\pi}{2}$ with the real axis, and A rotates such v by $\arg a$ and then scales the result by $|a| - |b|$. This completes the proof. □

As one always likes to do in mathematics, we can now pose the following inverse problem: given $\mu \in \mathrm{Bel}(V)$, how do we find an orientation-preserving $A : V \to W$ with $\mu(A) = \mu$?

Definition 1.5 (The Beltrami equation)**.** Given an almost-complex vector space V and $\mu \in \mathrm{Bel}(V)$, an orientation-preserving map $A : V \to W$ into some other almost-complex vector space W *satisfies the Beltrami equation for* μ if

$$\mu(A) = \mu.$$

△

Example 1.6 (Solving the Beltrami equation on $\mathbb{C}$)**.** Any $\mu \in \mathrm{Bel}(\mathbb{C})$ is of the form $\mu(z) = c\bar{z}$ for $c \in \mathbb{C}$, $|c| < 1$. Then the map $A : \mathbb{C} \to \mathbb{C}$, $z \mapsto z + c\bar{z}$ satisfies the Beltrami equation for μ:

$$A'(z) = z, \quad A''(z) = c\bar{z} = \mu(z), \quad ((A')^{-1} \circ A'')(z) = \mu(z).$$

Solutions to the Beltrami equation are unique up to postcomposition with an isomorphism $\mathbb{C} \to \mathbb{C}$ that commutes with i (i.e. multiplication by a scalar in $\mathbb{C}^*$): Given two solutions A, B to the Beltrami equation for μ, we have $A(z) =$

$a_1 z + a_2\overline{z}$ and $B(z) = b_1 z + b_2\overline{z}$ with $a_2/a_1 = b_2/b_1 = c$. It is straightforward to see that

$$B^{-1}(z) = \frac{1}{|b_2|^2 - |b_1|^2}\left(-\overline{b}_1 z + b_2\overline{z}\right)$$

and so

$$(A \circ B^{-1})(z) = \frac{1}{|b_2|^2 - |b_1|^2}\left((a_2\overline{b}_2 - a_1\overline{b}_1)z + (a_1 b_2 - a_2 b_1)\overline{z}\right).$$

We are done if we can show that the coefficient of $\overline{z}$ above is equal to 0. Dividing by $a_1 \neq 0$, we have

$$\begin{aligned}\frac{1}{a_1}(a_1 b_2 - a_2 b_1) &= b_2 - \frac{a_2}{a_1} b_1 \\ &= b_2 - \frac{b_2}{b_1} b_1 \\ &= 0.\end{aligned}$$

We conclude that $\mu : GL^+(2,\mathbb{R}) \to \mathrm{Bel}(\mathbb{C})$, $A \mapsto \mu(A)$ descends to a diffeomorphism

$$GL(1,\mathbb{C})\backslash GL^+(2,\mathbb{R}) = SO(2)\backslash SL(2,\mathbb{R}) \xrightarrow{\mu} \mathrm{Bel}(\mathbb{C}) \cong \mathbb{D}.$$

△

Theorem 1.7 (Geometric interpretation of the Beltrami equation)**.** *Let V be an almost-complex vector space, and let $\mu \in \mathrm{Bel}(V)$. Then μ has real and opposite eigenvalues. Let E be an ellipse in V whose major axis is the eigenspace for the positive eigenvalue of μ and whose ratio of major axis to minor axis is $\frac{1+|\mu|}{1-|\mu|}$. Then $A : V \to W$ solves the Beltrami equation for μ if and only if A maps E to a circle in F.*

Proof. Again we choose coordinates so that, without loss of generality, $V = W = \mathbb{C}$. Then $\mu(z) = c\overline{z}$ for $c \in \mathbb{C}$, $|c| < 1$. Therefore μ has eigenvalues $|c|$ and $-|c|$, whose eigenspaces are the lines making angles of $\frac{1}{2}\arg c$ and $\frac{1}{2}\arg c + \frac{\pi}{2}$, respectively, with the real axis. By Example 1.6, A solves the Beltrami equation for μ if and only if $A(z) = \alpha(z + c\overline{z})$ for some $\alpha \in \mathbb{C}^*$. By the proof of Lemma 1.4, we see that A acts on the $|c|$-eigenspace by rotation by $\arg\alpha$ and scaling by $|\alpha|(1 + |c|)$, and A acts on the $-|c|$-eigenspace by rotation by $\arg\alpha$ and scaling by $|\alpha|(1 - |c|)$. Therefore A takes E to a circle. Conversely, any map taking E to a circle must be of the form $z \mapsto \alpha(z + c\overline{z})$ for some $\alpha \in \mathbb{C}^*$, and so we are done. □

The last thing we need to talk about in the "infinitesimal" world of one almost-complex vector space at a time is the *pullback* of Beltrami forms. One might expect that since $\mu \in \mathrm{Bel}(V)$ is a map $V \to V$, we might be able to pull back μ along some orientation-preserving isomorphism $\varphi : W \to V$ via $(W \xrightarrow{\varphi^*\mu} W) := (W \xrightarrow{\varphi} V \xrightarrow{\mu} V \xrightarrow{\varphi^{-1}} W)$. This does not work. If $\varphi^*\mu$ is defined in this way, then it is not a Beltrami form: If $\mu \in \text{Bel}(\mathbb{C})$ and $\varphi : \mathbb{C} \to \mathbb{C}$ are given by $\mu(z) = c\overline{z}$ and $\varphi(z) = az + b\overline{z}$, then we have

$$(\varphi^{-1} \circ \mu \circ \varphi)(z) = \frac{1}{|b|^2 - |a|^2} \left(-2\text{Im}(ab\overline{c})z + c(b^2 - \overline{a}^2)\overline{z} \right).$$

In order for $(\varphi^{-1} \circ \mu \circ \varphi)$ to be a Beltrami form, the coefficient of z above must vanish. But this means that $ab\overline{c}$ is real, which is of course not always the case for arbitrary a, b.

Another attempt at defining the pullback of a Beltrami form is the following. Since every almost-complex vector space V is isomorphic to $\mathbb{C}$ as an almost-complex vector space, Example 1.6 shows that the Beltrami equation for $\mu \in \text{Bel}(V)$ always has a set of solutions $A : V \to U$ for any almost-complex vector space U. For any orientation-preserving isomorphism $A : V \to U$,

$$[A] := \{(B, W) \mid W \text{ an almost-complex vector space, } B : V \xrightarrow{\sim} W \text{ orientation-preserving, } \mu(B) = \mu(A)\}.$$

That is, $[A]$ is the collection of *all* solutions to the Beltrami equation for $\mu(A)$. Then there is a one-to-one correspondence of $\text{Bel}(V)$ with the set of all $[A]$, given by $\mu = \mu(A) \leftrightarrow [A]$. This correspondence presents us with another natural-looking definition of pullback: just precompose the solution A to the Beltrami equation for μ with the map φ. Given $[A] \leftrightarrow \mu \in \text{Bel}(V)$ and an orientation-preserving isomorphism $\varphi : W \to V$, it seems natural to define $\varphi^*[A] = [A \circ \varphi]$. Therefore, for $\mu = \mu(A)$, we define $\varphi^*\mu = \varphi^*(\mu(A)) = \mu(A \circ \varphi)$. In words, $\varphi^*\mu \in \text{Bel}(W)$ is the unique Beltrami form such that if A solves the Beltrami equation for μ, then $A \circ \varphi$ solves the Beltrami equation for $\varphi^*\mu$. This turns out to be the correct definition of pullback.

Definition 1.8 (Pullback of a Beltrami form)**.** Let V and W be almost-complex vector spaces, and let $\varphi : W \to V$ be an orientation-preserving isomorphism. We define the *pullback* $\varphi^* : \text{Bel}(V) \to \text{Bel}(W)$ along φ by

$$\varphi^*(\mu(A)) := \mu(A \circ \varphi).$$

△

Theorem 1.9 (Biholomorphicity of the pullback)**.** *Let V and W be almost-complex vector spaces, and let $\varphi : W \to V$ be an orientation-preserving isomorphism. Then $\varphi^* : \text{Bel}(V) \to \text{Bel}(W)$ is a biholomorphism with respect to the usual complex manifold structures on* $\text{Bel}(V)$ *and* $\text{Bel}(W)$.

Proof. It suffices to consider the case $V = W = \mathbb{C}$. Let $\mu \in \text{Bel}(\mathbb{C})$ and $\varphi : \mathbb{C} \to \mathbb{C}$ be given by $\mu(z) = c\overline{z}$ and $\varphi(z) = az + b\overline{z}$. Then $\mu = \mu(A)$ for $A : \mathbb{C} \to \mathbb{C}$ given by $A(z) = z + c\overline{z}$. We have $(A \circ \varphi)(z) = (a + c\overline{b})z + (b + c\overline{a})\overline{z}$. Recall that we have a biholomorphism $\text{Bel}(\mathbb{C}) \cong \mathbb{D}$ given by $(\mu(z) = c\overline{z}) \leftrightarrow c$. Since

$$\varphi^*\mu(z) = \mu(A \circ \varphi) = \frac{b + c\overline{a}}{a + c\overline{b}}\overline{z},$$

we conclude that under the identification $\mathrm{Bel}(\mathbb{C}) \cong \mathbb{D}$, the pullback φ^* is the conformal automorphism

$$c \mapsto \frac{b + c\overline{a}}{a + c\overline{b}}$$

of $\mathbb{D}$. □

1.2 The case of a Riemann surface

We return to the problem of measuring how much an orientation-preserving homeomorphism $f : X \to Y$ of Riemann surfaces differs from being a biholomorphism. As the preceding discussion suggests, we would like to use infinitesimal data. Of course, that requires f to have some well-defined notion of derivative. Therefore we must make a restriction on f beyond being an orientation-preserving homeomorphism. We will see that the following definition suits our needs perfectly.

Definition 1.10 (Definition 4.1.1 of [Hub06])**.** Let $U, V \subseteq \mathbb{C}$ be open, let $K \geq 1$, and let $k = (K-1)/(K+1)$, A homeomorphism $f : U \to V$ is K-*quasiconformal* if its distributional partial derivatives are locally L^2 functions and satisfy

$$\left|\frac{\partial f}{\partial \overline{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right|$$

almost everywhere. A map is *quasiconformal* if it is a K-quasiconformal homeomorphism for some K. △

Note that the inequality in the above definition can be restated as

$$|\mu(Df_p)| \leq k \quad \text{for a.e. } p \in U.$$

Note also that k is defined so that K is a global upper bound on the ratio of major axis to minor axis of the ellipses $Df_p(S^1)$ for $p \in U$ (see Theorem 1.7). As the preceding discussion suggests, we will use $\mu(Df_p)$ as our measurement of how much f differs from being a biholomorphism. Therefore we may think of a quasiconformal map as a function that differs from being a biholomorphism by a bounded amount. We summarize a variety of results about quasiconformal maps as follows. See [Hub06] or [IT92] for proofs of all of the claims the following proposition entails.

Proposition 1.11. *There is a well-defined category $\mathcal{Q}$ whose objects are Riemann surfaces and whose morphisms are homeomorphisms between Riemann surfaces that are quasiconformal when written in local coordinates. This category is in fact a groupoid.*

We call morphisms $f : X \to Y$ in $\mathcal{Q}$ *quasiconformal maps of Riemann surfaces.* Note that $\mu(Df_p)$ is well-defined at every $p \in X$, since any change of coordinates $h : U \to U$ on an open set U is holomorphic, and hence $\mu(Df_p) = \mu(D(f \circ h)_p)$.

For the sake of easy bookkeeping, we would like to organize all of the Beltrami forms $\mu(Df_p)$ into one object. We therefore give the following definition.

Definition 1.12. Let X be a Riemann surface. An L^∞ $\mathbb{R}$-linear bundle map $\mu : T_\mathbb{R}X \to T_\mathbb{R}X$ satisfying $\mu \circ J_X = -J_X \circ \mu$ and $\|\mu\|_\infty < 1$ is called a *Beltrami differential* on X. The space of all Beltrami differentials on X is denoted $\operatorname{Bel}(X)$, and has the structure of a Banach manifold: it is the open unit ball in the complex Banach space of L^∞ $\mathbb{R}$-linear bundle maps $\mu : T_\mathbb{R}X \to T_\mathbb{R}X$ satisfying $\mu \circ J_X = -J_X \circ \mu$.

For a quasiconformal map $f : X \to Y$ of Riemann surfaces, let $\mu(f) \in \operatorname{Bel}(X)$ be given by $\mu(f)_p := \mu(Df_p)$ at every $p \in X$. △

It is convenient to identify $\operatorname{Bel}(X)$ with the space of L^∞ sections of $\overline{T^*X} \otimes_\mathbb{C} TX$ that are essentially bounded by 1. We describe this identification as follows. Fix a point $p \in X$. Choosing a system of coordinates, we have the form $d\overline{z} : (T_\mathbb{C}X)_p \to \mathbb{C}$ given by $\frac{\partial}{\partial x} \mapsto 1$ and $\frac{\partial}{\partial y} \mapsto -i$. By restriction of scalars, we have an induced $\mathbb{R}$-linear map $d\overline{z} : (T_\mathbb{R}X)_p \to \mathbb{C}$ given by $\frac{\partial}{\partial x} \mapsto 1$ and $\frac{\partial}{\partial y} \mapsto -i$. Note that $d\overline{z} : (T_\mathbb{R}X)_p \to \mathbb{C}$ is anti-equivariant with respect to the almost-complex structures $((T_\mathbb{R}X)_p, J_X)$ and $(\mathbb{C}, i)$. Since $(T_\mathbb{R}X)_p$ and $\mathbb{C}$ are both 1-complex-dimensional, every $\mathbb{R}$-linear map $(T_\mathbb{R}X)_p \to \mathbb{C}$ that is anti-equivariant with respect to the almost-complex structures is of the form $c\,d\overline{z}$ for $c \in \mathbb{C}$.

Globally, this means that the complex line bundle whose fiber at a point p is the space of anti-equivariant $\mathbb{R}$-linear maps $(T_\mathbb{R}X)_p \to \mathbb{C}$ is isomorphic to the bundle $\overline{T^*X}$ (one must check that the transition maps are the right ones, but this is easy). Therefore the bundle B whose fiber at a point p is the space of $\mathbb{R}$-linear maps $\mu_p : (T_\mathbb{R}X)_p \to (T_\mathbb{R}X)_p$ satisfying $\mu_p \circ J_X = -J_X \circ \mu_p$ is isomorphic to the bundle $\overline{T^*X} \otimes_\mathbb{C} T_\mathbb{R}X$. By Appendix A.1, we have an isomorphism $T_\mathbb{R}X \cong TX$ of complex vector bundles, and so $B \cong \overline{T^*X} \otimes_\mathbb{C} TX$. Since, by definition, $\operatorname{Bel}(X)$ is the space of L^∞ sections of B that are essentially bounded by 1, we conclude that $\operatorname{Bel}(X)$ can also be identified with the space of L^∞ sections of $\overline{T^*X} \otimes_\mathbb{C} TX$ that are essentially bounded by 1. This means that in local coordinates on a coordinate chart $U \subseteq X$, any $\mu \in \operatorname{Bel}(X)$ may be written as

$$\mu(z) = \mu_U(z) d\overline{z} \otimes \frac{\partial}{\partial z}, \quad \mu_U \in L^\infty(U, \mathbb{C}).$$

We adopt the notation

$$\frac{d\overline{z}}{dz} := d\overline{z} \otimes \frac{\partial}{\partial z},$$

so that we can more succinctly write $\mu = \mu_U \frac{d\overline{z}}{dz}$. We will interchangeably think of a Beltrami differential as a bundle map and as a section of $\overline{T^*X} \otimes_\mathbb{C} TX$.

Again we have the inverse problem: given $\mu \in \operatorname{Bel}(X)$, how do we find a quasiconformal $f : X \to Y$ with $\mu(f) = \mu$? This is the *Beltrami equation for* μ. We first answer this question locally. The answer is exactly what you would guess after seeing Example 1.6.

Theorem 1.13 (Measurable Riemann mapping theorem, version 1)**.** *Let U be an open subset of $\mathbb{C}$, and let $\mu \in \operatorname{Bel}(U)$. Then there exists a quasiconformal map $f : U \to \mathbb{C}$ such that $\mu(f) = \mu$. Every other quasiconformal $g : U \to \mathbb{C}$ satisfying $\mu(g) = \mu$ is of the form $g = h \circ f$, where $h : f(U) \to \mathbb{C}$ is an injective analytic map (i.e. a biholomorphism onto its image).*

Proof. See [Hub06], Theorem 4.6.1. □

Now, let X be a Riemann surface, $\mu \in \mathrm{Bel}(X)$, and let $\{U_\alpha\}_\alpha$ be a covering of X by coordinate charts. Then by Theorem 1.13, for each α, there exists a quasiconformal map $f_\alpha : U_\alpha \to \mathbb{C}$ satisfying $\mu(f_\alpha) = \mu|_{U_\alpha}$. On the intersections $U_\alpha \cap U_\beta$, there is an injective analytic map $h : f_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}$ such that $f_\beta = h \circ f_\alpha$. That is to say, the transition function $f_\beta \circ (f_\alpha)^{-1}$ is analytic. Therefore we have a new complex structure $\{(U_\alpha, f_\alpha)\}_\alpha$ on X. Denote by X_μ the topological surface X endowed with this new complex structure. Note that X_μ indeed depends only on μ: any other choice of solution g_α the the Beltrami equation for μ on U_α is of the form $q \circ f_\alpha$ for $q : f_\alpha(U_\alpha) \to \mathbb{C}$ injective and analytic. Since q is a biholomorphism onto its image, using g_α instead of f_α gives the same complex structure X_μ.

Note that this setup also produces for us a quasiconformal map $f : X \to X_\mu$ of Riemann surfaces, simply by letting f be the identity map $X \to X$ on the underlying topological surface. Observe that when we write f down in coordinates on U_α, we simply get $f = f_\alpha$. Therefore $\mu(f) = \mu$. Furthermore, if $g : X \to X_\mu$ is another quasiconformal map of Riemann surfaces with $\mu(g) = \mu$, then Theorem 1.13 gives us that $g \circ f^{-1}$ is holomorphic, and hence $g = h \circ f$, where $h : X_\mu \to X_\mu$ is a biholomorphism. We summarize the preceding discussion as follows.

Theorem 1.14 (Measurable Riemann mapping theorem, version 2)**.** *Let X be a Riemann surface, and let $\mu \in \mathrm{Bel}(X)$. Then there exists a unique Riemann surface X_μ (up to biholomorphism) and a quasiconformal map $f : X \to X_\mu$ such that $\mu(f) = \mu$. Every other quasiconformal $g : X \to X_\mu$ satisfying $\mu(g) = \mu$ is of the form $g = h \circ f$, where h is a conformal automorphism of the Riemann surface X_μ.*

Finally, the notion of pullback of a Beltrami form induces a notion of pullback for Beltrami differentials.

Definition 1.15 (Pullback of a Beltrami differential)**.** Let X and Y be Riemann surfaces, and let $f : Y \to X$ be a quasiconformal map. We define the *pullback* $f^* : \mathrm{Bel}(X) \to \mathrm{Bel}(Y)$ along f by

$$f^*(\mu(g)) := \mu(g \circ f).$$

△

Theorem 1.16 (Biholomorphicity of the pullback)**.** *Let X and Y be Riemann surfaces, and let $f : Y \to X$ be a quasiconformal map. Then $f^* : \mathrm{Bel}(X) \to \mathrm{Bel}(Y)$ is a biholomorphism with respect to the complex Banach manifold structures on $\mathrm{Bel}(X)$ and $\mathrm{Bel}(Y)$.*

Proof. Similar to the proof of Theorem 1.9. See [Hub06], Proposition 4.8.17. □

1.3 Moduli of Riemann surfaces

The measurable Riemann mapping theorem (Theorem 1.14) tells us that we should guess that the moduli space of all Riemann surfaces quasiconformally equivalent to a given Riemann surface X is $\mathrm{Bel}(X)$, or perhaps a quotient of $\mathrm{Bel}(X)$. (Recall that by Proposition 1.11, admitting a quasiconformal map $X \to Y$ is an equivalence relation.) This makes more precise the informal discussion at the beginning of this section. In the rest of this section, all Riemann surfaces will be assumed to be compact.

So far, we have been discussing Riemann surfaces in generality, but the assumption that our surfaces are compact will imply that the resulting moduli space depends only on the topological type of the chosen surface, not on its complex structure. To be precise, we have the following theorem.

Theorem 1.17. *If two compact Riemann surfaces X and Y are homeomorphic, then they are quasiconformally equivalent.*

Proof. By the classification of compact surfaces, any two surfaces that are homeomorphic are also diffeomorphic by some diffeomorphism $f : X \to Y$. Every compact surface has an orientation-reversing involution, and so we may assume that f is orientation-preserving. Orientation-preserving diffeomorphisms of compact Riemann surfaces are clearly quasiconformal: if $f : X \to Y$ is such a diffeomorphism, then the function $X \to \mathbb{R}$ given by $p \mapsto |\mu(Df_p)|$ is bounded because X is compact. □

Theorem 1.17 says that for each $g \geq 0$, there is a connected component $\mathcal{Q}_g$ of the groupoid $\mathcal{Q}$ consisting of all Riemann surfaces of genus g. Note that given any closed oriented smooth surface S of genus g, there exists for each $X \in \mathcal{Q}_g$ an atlas of charts on S compatible with the smooth structure and orientation on S that induces a complex structure biholomorphic to X. Moreover, given any Riemann surface Y, a map $S \to Y$ is quasiconformal with respect to one such atlas of charts if and only if it is quasiconformal with respect to every such atlas; we call any such map *quasiconformal*. Indeed, if we think of Y as having S as its underlying smooth manifold, then we have the following definition.

Definition 1.18. Let S be a closed oriented smooth surface. We say a map $f : S \to S$ is *quasiconformal* if there exist atlases of charts $\{U_\alpha\}_\alpha$, $\{V_\beta\}_\beta$ on S that define complex structures on S compatible with the smooth structure and orientation on S such that f is quasiconformal with respect to the induced complex structures. Denote by $\mathrm{QC}(S)$ the topological group of all such maps, topologized with the compact-open topology.

Let $\mathrm{QC}_0(S)$ denote the subgroup of $\mathrm{QC}(S)$ of quasiconformal homeomorphisms $\varphi : S \to S$ that are homotopic to the identity $\mathrm{Id} : S \to S$. Note that $\mathrm{QC}_0(S)$ is a normal subgroup of $\mathrm{QC}(S)$.

The *mapping class group* $\mathrm{Mod}(S)$ is the quotient $\mathrm{QC}(S)/\mathrm{QC}_0(S)$. Note that $\mathrm{Mod}(S)$ is a discrete group. See Appendix B for further remarks on the definition of $\mathrm{Mod}(S)$. △

We can consider a smooth oriented surface S of genus g to be representative of $\mathcal{Q}_g$. In the following definition, we fix a collection of such representatives.

Definition 1.19. Let S_g denote some fixed closed smooth surface of genus $g \geq 0$ with some fixed orientation. Let $\overline{S_g}$ denote the same smooth surface endowed with the opposite orientation. Note that there exists an orientation-preserving diffeomorphism $S_g \to \overline{S_g}$, but no such diffeomorphism is homotopic to the identity map of the underlying smooth surface. △

The following proposition makes more precise the claim that S_g is "representative" of $\mathcal{Q}_g$.

Proposition 1.20. *Let $\mathscr{S}_g$ denote the groupoid whose only object is S_g (so $\mathscr{S}_g$ is a group in the categorical sense), and whose set of morphisms is $\mathrm{QC}(S_g)$. Then $\mathscr{S}_g$ and $\mathcal{Q}_g$ are equivalent categories.*

Proof. Succinctly, this holds because $\mathscr{S}_g$ is isomorphic to each subcategory $(X, \mathrm{Aut}_{\mathcal{Q}_g}(X))$ of $\mathcal{Q}_g$, and since $\mathcal{Q}_g$ is a groupoid, each such subcategory is a skeleton for $\mathcal{Q}_g$. We give a full proof below.

For each $X \in \mathcal{Q}_g$, choose an atlas of charts $\{U_\alpha^X\}_\alpha$ on S_g compatible with the smooth structure and orientation on S_g such that there is a biholomorphism $f^X : X \to S_g$ from X to S_g endowed with this atlas. Define a functor $\mathcal{F} : \mathcal{Q}_g \to \mathscr{S}_g$ on objects by $\mathcal{F}(X) = S_g$, and on morphisms by

$$\mathcal{F}(\varphi : X \to Y) = f^Y \circ \varphi \circ (f^X)^{-1}.$$

Define a functor $\mathcal{G} : \mathscr{S}_g \to \mathcal{Q}_g$ by picking an arbitrary $Y \in \mathcal{Q}_g$, and defining $\mathcal{G}(S_g) = Y$, and

$$\mathcal{G}(\psi : S_g \to S_g) = (f^Y)^{-1} \circ \psi \circ f^Y, \qquad \forall \psi \in \mathrm{QC}(S_g).$$

The composition $\mathcal{F} \circ \mathcal{G}$ is in fact equal to the identity functor on $\mathcal{S}_g$. We define a natural isomorphism $\eta : \mathcal{G} \circ \mathcal{F} \Rightarrow \mathrm{Id}_{\mathcal{Q}_g}$ by setting

$$\eta_X = (f^X)^{-1} \circ f^Y : \mathcal{G} \circ \mathcal{F}(X) = Y \to X.$$

This is obviously a natural isomorphism, and so we are done. □

To reflect the fact that our moduli space depends only on the topology of our surface, we define a space of Beltrami differentials for S_g.

Definition 1.21. Let $\mathrm{Bel}(S_g)$ denote the space of all equivalence classes of pairs $(\varphi : S_g \to X, \mu)$, where φ is a quasiconformal map from S_g to a Riemann surface X, and $\mu \in \mathrm{Bel}(X)$. Two pairs $(\varphi_1 : S_g \to X_1, \mu_1)$ and $(\varphi_2 : S_g \to X_2, \mu_2)$ are equivalent if $\mu_2 = (\varphi_1 \circ \varphi_2^{-1})^* \mu_1$. We denote the equivalence class of a pair (φ, μ) by $[\varphi, \mu]$. △

Remark 1.22. It is not hard to see that any quasiconformal map $\varphi : S_g \to X$ from S to a Riemann surface X induces a bijection

$$\begin{aligned} \varphi^* : \mathrm{Bel}(X) &\to \mathrm{Bel}(S_g) \\ \mu &\mapsto [\varphi, \mu], \end{aligned}$$

and so $\mathrm{Bel}(S_g)$ admits the structure of a complex Banach manifold biholomorphic to $\mathrm{Bel}(X)$ for any Riemann surface $X \in \mathcal{Q}_g$. △

Our guess that the moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus g is a quotient of $\mathrm{Bel}(S_g)$ is correct. In fact, it is the quotient by a group action. The group $\mathrm{QC}(S_g)$ acts on $\mathrm{Bel}(S_g)$ on the right in a straightforward way: given $f \in \mathrm{QC}(S_g)$ and $[\varphi, \mu] \in \mathrm{Bel}(S_g)$, we have

$$[\varphi, \mu].f \coloneqq [\varphi \circ f, \mu].$$

We can therefore make the following definitions.

Definition 1.23 (Teichmüller space and moduli space)**.** The *Teichmüller space* of S_g is the space $\mathrm{Teich}(S_g) \coloneqq \mathrm{Bel}(S_g)/\,\mathrm{QC}_0(S_g)$ with the quotient topology, and the *moduli space* of S_g is the space $\mathcal{M}_g \coloneqq \mathrm{Bel}(S_g)/\,\mathrm{QC}(S_g) = \mathrm{Teich}(S_g)/\,\mathrm{Mod}(S_g)$ with the quotient topology. △

Remark 1.24. Note that $\mathcal{M}_g$ is the quotient of $\mathrm{Teich}(S_g)$ by a discrete group. Though the action of $\mathrm{Mod}(S_g)$ on $\mathrm{Teich}(S_g)$ is not free, it is a theorem of Fricke that this action is properly discontinuous. We will later see that $\mathrm{Teich}(S_g)$ is a finite-dimensional contractible manifold. Along with Fricke's theorem, this implies that $\mathrm{Teich}(S_g)$ is the *orbifold universal cover* of $\mathcal{M}_g$. △

While Definition 1.23 is concise and clean and showcases the relationship between $\mathrm{Bel}(S_g)$, $\mathrm{Teich}(S_g)$, and $\mathcal{M}_g$, it would be preferable to have more direct descriptions of the points of $\mathrm{Teich}(S_g)$ and $\mathcal{M}_g$ than simply as orbits of elements of $\mathrm{Bel}(S_g)$ under a group action. We give these descriptions now. First observe that if $[\varphi : S \to X, \mu] \in \mathrm{Bel}(S_g)$, then if we let $g : X \to X_\mu$ be such that $\mu(g) = \mu$, then $[\varphi, \mu] = [g \circ \varphi : S \to X_\mu, 0]$. Therefore every element of $\mathrm{Bel}(S_g)$ can be written in the form $[\psi : S \to Y, 0]$.

Proposition 1.25. *Two elements* $[\varphi_1 : S \to X_1, 0], [\varphi_2 : S \to X_2, 0] \in \mathrm{Bel}(S_g)$ *lie in the same orbit of* $\mathrm{QC}_0(S_g)$ *if and only if there exists a biholomorphism* $j : X_2 \to X_1$ *such that* $j \circ \varphi_2$ *is homotopic to* φ_1*.*

Proof. Suppose $[\varphi_1, 0]$ and $[\varphi_2, 0]$ lie in the same orbit, so that there exists some quasiconformal $f : S_g \to S_g$ homotopic to the identity such that $[\varphi_1 \circ f, 0] = [\varphi_2, 0]$. By definition, this equality means that $(\varphi_1 \circ f \circ \varphi_2^{-1})^* 0 = 0$, which is to say that $j \coloneqq \varphi_1 \circ f \circ \varphi_2^{-1} : X_2 \to X_1$ is a biholomorphism. Since f is homotopic to the identity, $j \circ \varphi_2$ is homotopic to φ_1.

Now suppose there exists such a j. Letting $f \coloneqq \varphi_1^{-1} \circ j \circ \varphi_2$, we see that $[\varphi_1, 0].f = [\varphi_2, 0]$. □

Corollary 1.26.

$$\mathrm{Teich}(S_g) = \frac{\{\varphi : S_g \to X \mid X \text{ a Riemann surface, } \varphi \text{ quasiconformal}\}}{(\varphi_1 : S_g \to X_1) \sim (\varphi_2 : S_g \to X_2) \iff \exists h : X_2 \to X_1 \text{ a biholomorphism with } h \circ \varphi_2 \simeq \varphi_1}.$$

One also often sees $\mathrm{Teich}(S_g)$ and $\mathrm{Mod}(S_g)$ defined not in terms of quasiconformal maps, but in terms of orientation-preserving diffeomorphisms. By Corollary B.6, we have $\mathrm{Mod}(S_g) = \pi_0 \mathrm{Diff}^+(S_g)$. Furthermore, Lemma B.2 tells us that every homotopy class of quasiconformal maps has a representative that is an orientation-preserving diffeomorphism, and hence we have the following.

Corollary 1.27.

$$\mathrm{Teich}(S_g) = \frac{\{\varphi : S_g \to X \mid X \text{ a Riemann surface, } \varphi \text{ an orientation-preserving diffeomorphism}\}}{(\varphi_1 : S_g \to X_1) \sim (\varphi_2 : S_g \to X_2) \iff \exists h : X_2 \to X_1 \text{ a biholomorphism with } h \circ \varphi_2 \simeq \varphi_1}.$$

Finally, we obtain a characterization of $\mathcal{M}_g$ that makes no reference to general quasiconformal or smooth maps, only to biholomorphisms.

Proposition 1.28. *Two elements* $[\varphi_1 : S \to X_1, 0], [\varphi_2 : S \to X_2, 0] \in \mathrm{Bel}(S_g)$ *lie in the same orbit of* $\mathrm{QC}(S_g)$ *if and only if* X_1 *is biholomorphic to* X_2*.*

Proof. Suppose $[\varphi_1, 0]$ and $[\varphi_2, 0]$ lie in the same orbit, so that there exists some quasiconformal $f : S_g \to S_g$ such that $[\varphi_1 \circ f, 0] = [\varphi_2, 0]$. By definition, this equality means that $(\varphi_1 \circ f \circ \varphi_2^{-1})^* 0 = 0$, which is to say that $j \coloneqq \varphi_1 \circ f \circ \varphi_2^{-1} : X_2 \to X_1$ is a biholomorphism.

Now suppose there exists a biholomorphism $j : X_2 \to X_1$. Letting $f \coloneqq \varphi_1^{-1} \circ j \circ \varphi_2$, we see that $[\varphi_1, 0].f = [\varphi_2, 0]$. □

Corollary 1.29.

$$\mathcal{M}_g = \frac{\{X \text{ a Riemann surface} \mid X \text{ has genus } g\}}{X \sim Y \iff X \text{ is biholomorphic to } Y}$$

Corollaries 1.27 and 1.29 are usually given as the *definitions* of $\mathrm{Teich}(S_g)$ and $\mathcal{M}_g$. Note that these definitions do not immediately induce a topology on either space, but our Definition 1.23 does. There is another common definition of Teichmüller space in terms of representations of $\pi_1(S_g)$ that also induces a topology; see Theorem 3.1. That approach to Teichmüller theory is very important, but to pursue it now would mean putting complex analysis on hold for a while, so we will not go down that route at the moment.

2 Holomorphic Quadratic Differentials

Just as we came up with certain tensors (Beltrami differentials) that measured the difference between two complex structures in the last section, in this section we will come up with tensors (holomorphic quadratic differentials) that measure the difference between two *projective structures*.

Definition 2.1 (Projective structure)**.** Let S be a topological surface. A *projective structure* on S is a system of coordinate charts $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{CP}^1)\}_\alpha$ such that the coordinate transition functions are Möbius transformations, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} \in \mathrm{PSL}_2(\mathbb{C})$. (Note that, technically, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is only defined on the image of φ_α, but if it is expressible as a fractional linear transformation, then it extends to all of $\mathbb{CP}^1$, and so there is nothing wrong with writing $\varphi_\beta \circ \varphi_\alpha^{-1} \in \mathrm{PSL}_2(\mathbb{C})$.)

We call a surface endowed with a complex projective structure a *complex projective surface.* △

2.1 The Schwarzian derivative at a point

Just as a Beltrami differential measures how much a quasiconformal map differs from a biholomorphism, a Schwarzian derivative measures how much a holomorphic map differs from a Möbius transformation. We first give some linear-algebraic preliminaries.

Definition 2.2. Just as a *linear map* $V \to W$ of vector spaces may be defined as a mapping given by homogeneous linear polynomials in every system of coordinates (i.e. choice of basis), a *degree k map* (*quadratic map*, *cubic map*, etc.) $V \to W$ of vector spaces is a mapping given by homogeneous degree k polynomials in every system of coordinates. Let $\mathrm{Hom}^k(V, W)$ denote the vector space of degree k maps $V \to W$. Let F denote the field over which the vector space V is defined; elements of $\mathrm{Hom}^k(V, F)$ are called *degree k forms* (*quadratic forms*, *cubic forms*, etc.) △

Example 2.3. The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x^3 - xy^2, 10y^3)$ is a cubic map of vector spaces: If we precompose T with an invertible linear transformation (i.e. a change of coordinates) $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, the result is still given by homogeneous cubic polynomials

$$T \circ A(x, y) = (2(a_1x + a_2y)^3 - (a_1x + a_2y)(a_3x + a_4y)^2, 10(a_3x + a_4y)^3).$$

Similarly, if we postcompose T with a change of coordinates $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$, the result is still given by homogeneous cubic polynomials

$$B \circ T(x, y) = (b_1(2x^3 - xy^2) + b_2(10y^3), b_3(2x^3 - xy^2) + b_4(10y^3)).$$

△

Remark 2.4. Observe that $\mathrm{Hom}^k(V, W)$ is isomorphic to $\mathrm{Sym}^k(V^*) \otimes W$ via the natural isomorphism defined on simple tensors by

$$\begin{aligned} \mathrm{Sym}^k(V^*) \otimes W &\xrightarrow{\sim} \mathrm{Hom}^k(V, W) \\ (\alpha_1 \cdots \alpha_k) \otimes w &\mapsto \left(v \mapsto \left(\prod_{i=1}^k \alpha_i(v) \right) w \right). \end{aligned}$$

Note that in the case $k = 1$, this is just the ordinary isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$. △

Lemma 2.5. *Let V be a 1-dimensional vector space over a field F, and let $k, \ell \in \mathbb{N}$ with $k \geq \ell$. Then there is an isomorphism $\Psi : \mathrm{Hom}^{k-\ell}(V, F) \to \mathrm{Hom}^k(V, V^{\otimes \ell})$ given by*

$$\varphi \mapsto \left(v \mapsto \varphi(v) v^{\otimes \ell} \right).$$

Proof. By Remark 2.4, it is clear that the vector spaces $\mathrm{Hom}^{k-\ell}(V, F)$ and $\mathrm{Hom}^k(V, V^{\otimes \ell})$ are 1-dimensional. Therefore, since the map Ψ is nonzero, it is an isomorphism. $\square$

The following lemma is an elementary fact about differential topology, though not one that is often found in textbooks. It is a special case of Principle 2.3.1 in [Hub06], which is given as an exercise in that text. The proof of the general principle is the same as the proof of the following lemma, but with more bookkeeping.

Lemma 2.6. *Let X and Y be Riemann surfaces, let $p \in X$, and let $f : X \to Y$ be a holomorphic function. Let $\varphi : U \to \mathbb{C}$ be a coordinate chart about p, and $\psi : V \to \mathbb{C}$ be a coordinate chart about $f(p)$, so that $\varphi(p) = \psi(f(p)) = 0$. Suppose that the Taylor series of $\psi \circ f \circ \varphi^{-1}$ has no terms of degree less than k for some $k \geq 1$. Then the same is true for any other choices of coordinate charts. Furthermore, if $\frac{a_k}{k!} z^k + \cdots$ is this Taylor series, then*

$$T_p X \underset{\varphi_*}{\cong} T_0\mathbb{C} = \mathbb{C} \to \mathbb{C} = T_0\mathbb{C} \underset{\psi_*}{\cong} T_{f(p)} Y$$
$$z \mapsto a_k z^k$$

is a degree k map $D_p^k f : T_p X \to T_{f(p)} Y$ of vector spaces that depends only on f, not on the choices of coordinate charts.

Remark 2.7. Observe that in Lemma 2.6, the map $D_p^1 f$ is just the ordinary derivative $Df = f_* : T_p X \to T_{f(p)} Y$. The map $D_p^k f$ can be thought of as the kth derivative of f at p. Then Lemma 2.6 states that the kth derivative of f at p is independent of the choice of coordinates when f vanishes to degree $k-1$ (cf. the Hessian of a scalar-valued function on a manifold, which is only independent of coordinates at critical points of the function). $\triangle$

Proof. Let $\varphi' : U' \to \mathbb{C}$, $\psi' : V' \to \mathbb{C}$ be other choices of coordinate charts with $\varphi'(p) = \psi'(f(p)) = 0$. For ease of notation, let $g := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$, let $\Phi := \varphi \circ (\varphi')^{-1} : \varphi'(U \cap U') \to \varphi(U \cap U')$, and let $\Psi := \psi' \circ \psi^{-1} : \psi(V \cap V') \to \psi'(V \cap V')$. Then $\frac{a_k}{k!} z^k + \cdots$ is the Taylor series for g about 0; let

$$\Phi(z) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} z^n, \qquad \Psi(z) = \sum_{n=1}^{\infty} \frac{\beta_n}{n!} z^n$$

be the Taylor series for Φ and Ψ about 0. Note that by our assumptions, these Taylor series indeed do not have any degree zero terms. Now, the Taylor series for $\psi' \circ f \circ (\varphi')^{-1} = \Psi \circ g \circ \Phi$ about 0 is

$$\begin{aligned} \Psi \circ g \circ \Phi(z) &= \sum_{n=1}^{\infty} \frac{\beta_n}{n!} \left(\sum_{m=k}^{\infty} \frac{a_m}{m!} \left(\sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{\ell!} z^\ell \right)^m \right)^n \\ &= \frac{\beta_1 a_k \alpha_1^k}{k!} z^k + \text{ higher degree terms.} \end{aligned}$$

This proves the claim that f has trivial Taylor coefficients of order less than k in every system of coordinates.

The map $D^k_p f$ is certainly a well-defined degree k map of vector spaces; the question is whether it is independent of the choices of coordinate charts. We see that the coordinate charts φ', ψ' give us the degree k map

$$(D')^k_p f : T_p X \underset{\varphi'_*}{\cong} T_0\mathbb{C} = \mathbb{C} \to \mathbb{C} = T_0\mathbb{C} \underset{\psi'_*}{\cong} T_{f(p)} Y$$
$$z \mapsto \beta_1 a_k \alpha_1^k z^k.$$

We must show that $(D')^k_p f$ is in fact the same map as $D^k_p f$. To see this, we use the coordinate transition maps Φ and Ψ to write $(D')^k_p f$ in the coordinates given by φ and ψ:

$$\mathbb{C} \xrightarrow{\Phi_*^{-1}} \mathbb{C} \xrightarrow{\psi'_* \circ (D')^k_p f \circ (\varphi'_*)^{-1}} \mathbb{C} \xrightarrow{\Psi_*^{-1}} \mathbb{C}$$

$$z \longmapsto \alpha_1^{-1} z \longmapsto \beta_1 a_k \alpha_1^k \left(\alpha_1^{-1} z\right)^k = \beta_1 a_k z^k \longmapsto \beta_1^{-1}\left(\beta_1 a_k z^k\right) = a_k z^k.$$

The above map is $\psi_* \circ (D')^k_p f \circ (\varphi_*)^{-1}$, and we see that it is indeed equal to $\psi_* \circ D^k_p f \circ (\varphi_*)^{-1}$. Since φ_* and ψ_* are invertible, we conclude that $(D')^k_p f = D^k_p f$. □

The key ingredient in the definition of the Schwarzian derivative is the fact that every meromorphic function with nonvanishing first derivative can be approximated to second order by a Möbius transformation.

Lemma 2.8. *Let $U \subseteq \mathbb{CP}^1$ be open, let $f : U \to \mathbb{CP}^1$ be a holomorphic function with $Df \neq 0$ everywhere, and let $z_0 \in U$. Then there is a unique $A \in \mathrm{PSL}_2(\mathbb{C})$ with $\frac{\partial^k A}{\partial z^k}(z_0) = \frac{\partial^k f}{\partial z^k}(z_0)$ for all $0 \leq k \leq 2$.*

Proof. Since $\mathrm{PSL}_2(\mathbb{C})$ acts transitively on $\mathbb{CP}^1$ by biholomorphisms, we may assume without loss of generality that U contains 0, that $z_0 = 0$, and that $f(0) = 0$. Then if $\sum \frac{a_n}{n!} z^n$ is the Taylor series for f about 0, the map

$$A(z) = \frac{a_1 z}{1 - (a_2/2a_1)z}$$

is the desired Möbius transformation. □

We are now in the position to define the Schwarzian derivative of a holomorphic map $f : X \to Y$ of complex projective surfaces, where Df is assumed to be nontrivial everywhere. Let $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{CP}^1)\}_\alpha$ denote the projective structure on X, and let $\{(V_\beta, \psi_\beta : V_\beta \to \mathbb{CP}^1)\}_\beta$ denote the projective structure on Y. Let $p \in X$, let (U, φ) be one of the charts $(U_\alpha, \varphi_\alpha)$ about p, and let (V, ψ) be one of the charts (V_β, ψ_β) about $f(p)$. Suppose that $\psi(f(p)) = 0$. Then let $A_{f,p,\varphi,\psi} : \varphi(U) \to \mathbb{CP}^1$ be the unique Möbius transformation approximating $\psi \circ f \circ \varphi^{-1}$ to second order at 0.

Since $\psi(f(p)) = 0$, the difference $(\psi \circ f \circ \varphi^{-1}) - A_{f,p,\varphi,\psi}$ is well-defined in a neighborhood of 0 in $\varphi(U)$. Let us suppose that U is small enough that this difference is well-defined on all of $\varphi(U)$. If (U', φ') is some other coordinate chart about p, then the coordinate transition map $B = \varphi \circ (\varphi')^{-1}$ is a Möbius transformation, and so $A_{f,p,\varphi',\psi} = A_{f,p,\varphi,\psi} \circ B$. Therefore

$$(\psi \circ f \circ (\varphi')^{-1}) - A_{f,p,\varphi',\psi} = ((\psi \circ f \circ \varphi^{-1}) - A_{f,p,\varphi,\psi}) \circ (\varphi \circ (\varphi')^{-1}).$$

Similarly, if (V', ψ') is some other coordinate chart about $f(p)$ with $\psi'(f(p)) = 0$, then

$$(\psi' \circ f \circ \varphi^{-1}) - A_{f,p,\varphi,\psi'} = (\psi' \circ \psi^{-1}) \circ ((\psi \circ f \circ \varphi^{-1}) - A_{f,p,\varphi,\psi}).$$

Therefore the function

$$\begin{aligned} &(f - A)_p : U \to V \\ &(f - A)_p(z) \coloneqq \psi^{-1}(((\psi \circ f \circ \varphi^{-1}) - A_{f,p,\varphi,\psi}) \circ \varphi(z)), \qquad \forall z \in U \end{aligned}$$

is well-defined independent of the choice of coordinate functions φ and ψ. Since the Taylor series of the function $(f - A)_p$ has no terms of degree less than 2, Lemma 2.6 tells us that there is a well-defined cubic map

$$D^3_p(f - A)_p : T_pX \to T_{f(p)}Y.$$

Postcomposing with Df^{-1} gives us a cubic map

$$Df^{-1} \circ D^3_p(f - A)_p : T_pX \to T_pX,$$

and hence by Lemma 2.5 (with $k = 3$ and $\ell = 1$) a quadratic form

$$S(f)_p : T_pX \to \mathbb{C}.$$

Definition 2.9. Let $f : X \to Y$ be a holomorphic map of complex projective surfaces, where Df is nontrivial everywhere. Then the quadratic form $S(f)_p$ is called the *Schwarzian derivative* of f at $p \in X$. △

2.2 The Schwarzian derivative as a tensor

Just as we were able to fit together all of the $\mu(f)_p$ into an L^∞ section $\mu(f)$ of $\overline{T^*X} \otimes_{\mathbb{C}} TX$ for a quasiconformal map f, we are able to fit together all of the $S(f)_p$ into a holomorphic section $S(f)$ of $\mathrm{Sym}^2(T^*X) = T^*X \otimes_{\mathbb{C}} T^*X$ (note that this equality holds because X is 1-dimensional as a complex manifold). By Remark 2.4, $\mathrm{Sym}^2(T^*X) \cong \mathrm{Hom}^2(T^*X, F)$, and so holomorphic sections of $\mathrm{Sym}^2(T^*X)$ are called *holomorphic quadratic differentials* on X. Since $\mathrm{Sym}^2(T^*X)$ is a holomorphic vector bundle, it makes sense to drop the qualifier "holomorphic" and just call these sections *quadratic differentials* on X. Let $QD(X)$ denote the vector space of all quadratic differentials on X. It is a consequence of the Riemann-Roch theorem that when X is a closed Riemann surface of genus g, we have $\dim_{\mathbb{C}} QD(X) = 3g - 3$; see Appendix C.

Lemma 2.10. *Let $f : X \to Y$ be a holomorphic map of complex projective surfaces, where Df is nontrivial everywhere. Then the section $S(f) : p \mapsto S(f)_p$ of $\mathrm{Sym}^2(T^*X)$ is holomorphic, and hence $S(f)$ is a quadratic differential.*

Proof. First, a proof sketch: Looking at the formula in Lemma 2.8, we see that all the Taylor coefficients of the Möbius transformation that approximates f depend holomorphically (indeed, rationally) on the Taylor coefficients of f, which themselves vary holomorphically in p. Therefore $S(f)$ is a holomorphic map.

Explicitly, let $\varphi : U \to \mathbb{CP}^1$ and $\psi : U \to \mathbb{CP}^1$ be coordinate charts around p and $f(p)$, respectively, such that $\varphi(p) = \psi(\varphi(p)) = 0$. If $\sum \frac{a_n}{n!} z^n$ is the Taylor series for $g := \psi \circ f \circ \varphi^{-1}$ about 0, then the formula in Lemma 2.8 gives us that $A_{f,p,\varphi,\psi}(z) = a_1 z + a_2 z^2 + \frac{a_2^2}{4a_1} z^3 + \cdots$. Therefore, we have

$$\begin{aligned} D^3_p(f-A)_p : T_pX \underset{\varphi_*}{\cong} T_0\mathbb{C} = \mathbb{C} \to \mathbb{C} = T_0\mathbb{C} \underset{\psi_*}{\cong} T_{f(p)}Y \\ z \mapsto 6\left(\frac{a_3}{6} - \frac{a_2^2}{4a_1}\right) z^3, \end{aligned}$$

and hence

$$\begin{aligned} Df^{-1} \circ D^3_p(f-A)_p : T_pX \underset{\varphi_*}{\cong} T_0\mathbb{C} = \mathbb{C} \to \mathbb{C} = T_0\mathbb{C} \underset{\varphi_*}{\cong} T_pX \\ z \mapsto \frac{6}{a_1}\left(\frac{a_3}{6} - \frac{a_2^2}{4a_1}\right) z^3 \\ = \left(\frac{a_3}{a_1} - \frac{3}{2}\left(\frac{a_2}{a_1}\right)^2\right) z^3 \end{aligned}$$

Therefore, in the local coordinate w given by φ, the section $S(f) : X \to \mathrm{Sym}^2(T^*X)$ has the formula

$$S(f) = \left(\frac{\frac{\partial^3 g}{\partial w^3}(w)}{\frac{\partial g}{\partial w}(w)} - \frac{3}{2}\left(\frac{\frac{\partial^2 g}{\partial w^2}(w)}{\frac{\partial g}{\partial w}(w)}\right)^2\right) dw^2.$$

Since g is holomorphic, this formula shows that $S(f)$ is also holmorphic. □

Once again, we have an inverse problem: given $q \in QD(X)$, how do we find a holomorphic $f : X \to Y$ with $S(f) = q$? This is the *Schwarzian differential equation for q*. We will first answer this question locally.

Theorem 2.11 (Solving the Schwarzian equation, version 1)**.** *Let U be a simply-connected open subset of $\mathbb{CP}^1$, and let $q \in QD(U)$. Then there exists a holomorphic map $f : U \to \mathbb{CP}^1$ (i.e. a meromorphic function on U) such that $S(f) = q$. Every other holomorphic $g : U \to \mathbb{CP}^1$ satisfying $S(g) = q$ is of the form $g = A \circ f$, where $A \in \mathrm{PSL}_2(\mathbb{C})$.*

Proof. See [Hub06], Proposition 6.3.7. □

Similarly to our discussion after version 1 of the measurable Riemann mapping theorem (Theorem 1.13), we can patch together local solutions of the Schwarzian differential equation on a Riemann surface to get a global solution.

Let X be a Riemann surface with projective structure $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{CP}^1)\}_\alpha$, where each U_α is simply-connected, and let $q \in \mathrm{QD}(X)$. Then by Theorem 2.11, for each α, there exists a holomorphic map $f_\alpha : U_\alpha \to \mathbb{CP}^1$ satisfying $S(f_\alpha) = q|_{U_\alpha}$. For each intersection $U_\alpha \cap U_\beta$, there is a Möbius transformation $A_{\alpha\beta} \in \mathrm{PSL}_2(\mathbb{C})$ such that $f_\beta = A_{\alpha\beta} \circ f_\alpha$. That is to say, the transition function $f_\beta \circ (f_\alpha)^{-1}$ is a Möbius transformation. Therefore we have a new projective structure $\{(U_\alpha, f_\alpha)\}_\alpha$ on X. Denote by X_q the topological surface X endowed with this new projective structure. Note that X_q indeed depends only on q: any other choice of solution g_α to the Schwarzian differential equation for q on U_α is of the form $B \circ f_\alpha$ for $B \in \mathrm{PSL}_2(\mathbb{C})$ a Möbius transformation. Since B is a Möbius transformation, using g_α instead of f_α gives the same Riemann surface X_q with projective structure.

Note that this setup also produces for us a biholomorphism $f : X \to X_q$ of Riemann surfaces with projective structure, simply by letting f be the identity map $X \to X$ on the underlying topological surface. Observe that when we write f down in coordinates on U_α, we simply get $f = f_\alpha$. Therefore $S(f) = q$. Furthermore, if $g : X \to X_q$ is another holomorphic map with $S(g) = q$, then Theorem 2.11 gives us that $g \circ f^{-1}$ is a Möbius transformation, and hence $g = h \circ f$, where $h : X_q \to X_q$ is an automorphism of the projective structure on X_q (i.e. a biholomorphism $X_q \to X_q$ that is fractional-linear when written in local coordinates).

Note that the preceding discussion mirrors exactly the discussion after version 1 of the measurable Riemann mapping theorem. We summarize this discussion as follows.

Theorem 2.12 (Solving the Schwarzian equation, version 2)**.** *Let X be a complex projective surface, and let $q \in QD(X)$. Then there exists a unique complex projective surface X_q (up to isomorphism of complex projective surfaces) and a biholomorphism $f : X \to X_q$ such that $S(f) = q$. Every other biholomorphism $g : X \to X_q$ satisfying $S(g) = q$ is of the form $g = h \circ f$, where h is an automorphism of the projective structure on X_q.*

Remark 2.13. By the uniformization theorem, the group of projective automorphisms of a complex projective surface X is isomorphic to the group of biholomorphisms of X.

This section is meant to mirror §1.2, and so we should discuss the pullback of quadratic differentials. It seems there is nothing special to say here. Since a quadratic differential q on Y is a holomorphic section of $T^*Y \otimes_{\mathbb{C}} T^*Y$, we already have a notion of pullback of q along a holomorphic map $f : X \to Y$ given by

$$f^*(\alpha \otimes \beta) = f^*\alpha \otimes f^*\beta.$$

However, there is more than meets the eye. Whereas we had (by definition) $f^*\mu(g) = \mu(g \circ f)$, one can see that $f^*S(g) \neq S(g \circ f)$. Instead, given holomorphic maps $f : X \to Y$ and $g : Y \to Z$ of complex projective surfaces, we have

$$f^*S(g) = S(g \circ f) - S(f). \tag{2}$$

Equation 2 is called the *cocycle condition.*

2.3 Moduli of projective structures

For a far more in-depth look at moduli of projective structures, see [Dum09].

Definition 2.14. The set $\mathrm{MProj}(S_g)$ of *marked projective structures* is the collection of all equivalence classes of orientation-preserving diffeomorphisms $\varphi : S_g \to X$ from S_g to a complex projective surface X, where two maps $\varphi_1 : S_g \to X_1$ and $\varphi_2 : S_g \to X_2$ are equivalent if there is an isomorphism $h : X_2 \to X_1$ of projective structures such that $h \circ \varphi_2$ is homotopic to φ_1. The equivalence class of a map φ is denoted $[\varphi]$, and is called an *marked projective structure* on S.

Since every isomorphism of projective structures is a biholomorphism, it follows from Corollary 1.27 that we have a surjection $\pi : \mathrm{MProj}(S_g) \to \mathrm{Teich}(S_g)$ given by weakening the equivalence relation from requiring that h be an isomorphism of projective structures to requiring that h merely be some biholomorphism.

Remark 2.15. Note that, generically, given two closed Riemann surfaces X and Y, there exists at most one biholomorphism $X \to Y$. Therefore, the question of whether two complex projective surfaces X and Y are isomorphic is usually just the question "Are X and Y biholomorphic, and if so, does the unique biholomorphism respect the given projective structures?"

Lemma 2.16. *For every $\tau = [\varphi : S_g \to X] \in \mathrm{Teich}(S_g)$, the fiber $\pi^{-1}(\tau)$ is an affine space modeled on $\mathrm{QD}(X)$, where*

$$[\psi : S_g \to Y] + q = [f \circ \psi : S_g \to Y_q], \qquad [\psi] \in \pi^{-1}(\tau),\, q \in \mathrm{QD}(Y) \cong \mathrm{QD}(X),$$

where $f : Y \to Y_q$ is the unique holomorphic map with $S(f) = q$ such that there is a biholomorphism $h : Y_q \to X$ such that $h \circ f$ is homotopic to φ.

Proof. Given $q \in \mathrm{QD}(Y)$, Theorem 2.12 tells us that there exists a biholomorphism $f : Y \to Y_q$ with $S(f) = q$. By Remark 2.13 and the biholomorphicity of automorphisms of projective structure, we can always choose f so that there is a biholomorism $h : Y_q \to X$ such that $h \circ f$ is homotopic to φ. Therefore $[\psi] + q$ is well-defined for every $\psi \in \pi^{-1}(\tau)$ and every $q \in \mathrm{QD}(X)$.

To see that $[\psi] \mapsto [\psi] + q$ is indeed a group action, note first that we obviously have $[\psi] + 0 = [\psi]$. Now, given $q, r \in \mathrm{QD}(X)$, let $f : Y \to Y_q$ be the map associated to q, and let $g : Y_q \to (Y_q)_r$ be the map associated to r. The cocycle condition (2) implies that $q + r = S(g \circ f)$, and hence $([\psi] + q) + r = [\psi] + (q + r)$.

Therefore $[\psi] \mapsto [\psi] + q$ is indeed a group action of $\mathrm{QD}(X)$ on $\pi^{-1}(\tau)$. That this group action is free and transitive is another consequence of Theorem 2.12. $\square$

In Theorem 3.10 we will see that $\mathrm{Teich}(S_g)$ is a complex manifold whose holomorphic tangent space at $[\varphi : S_g \to X]$ is canonically identified with $\mathrm{QD}(X)$. Therefore, any global section of π induces a bijection $\mathrm{MProj}(S_g) \to T^* \mathrm{Teich}(S_g)$ that, for each $[\varphi : S_g \to X] \in \mathrm{Teich}(S_g)$, is equivariant with respect to the $\mathrm{QD}(X)$ action on the fiber over $[\varphi : S_g \to X]$. When $g \geq 2$, there is a preferred section of π called the *Fuchsian section*. Given a Riemann surface X of genus $g \geq 2$, the uniformization theorem states that there is a locally biholomorphic covering map $\mathbb{H}^2 \to X$ obtained by taking the quotient of $\mathbb{H}^2$ by a Fuchsian group. This covering endows X with the projective structure whose coordinate maps $U \to \mathbb{H}^2 \subseteq \mathbb{CP}^1$ are precisely the local sections of the covering. Let X_{Fuchs} denote the surface X endowed with this projective structure.

Definition 2.17. The *Fuchsian section* of $\pi : \mathrm{MProj}(S_g) \to \mathrm{Teich}(S_g)$ is the map

$$\begin{aligned} \sigma_{\mathrm{Fuchs}} : \mathrm{Teich}(S_g) &\to \mathrm{MProj}(S_g) \\ [\varphi : S_g \to X] &\mapsto [\varphi : S_g \to X_{\mathrm{Fuchs}}]. \end{aligned}$$

To complete the analogy between projective structures and complex structures, note that $\mathrm{Mod}(S_g)$ acts on $\mathrm{MProj}(S_g)$ in the same way as it acts on $\mathrm{Teich}(S_g)$, and so we can similarly form the quotient $\mathrm{MProj}(S_g)/\mathrm{Mod}(S_g)$, the space of unmarked projective structures.

3 The Bers Embedding

The stories of sections 1 and 2 appear to run parallel to each other, but in fact they intersect. Their point of intersection is the Bers embedding theorem. This gives $\mathrm{Teich}(S_g)$ the structure of a complex manifold whose tangent space at $[\varphi : S_g \to X]$ is naturally identified with a quotient of $\mathrm{Bel}(X)$, and whose cotangent space at $[\varphi : S_g \to X]$ is naturally identified with $\mathrm{QD}(X)$.

A very clean exposition of the material of this section can be found in [McM00]. From now on, we will be assuming $g \geq 2$.

3.1 Quasi-Fuchsian representations

Recall that a *Fuchsian* representation of a surface group is a discrete and faithful representation $\pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{R})$. The space $\mathcal{DF}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$ of all such representations is a subspace of $\mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$, and hence inherits the subspace topology from this Hom space, which is naturally endowed with the compact-open topology. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on representations $\rho \in \mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$ by conjugation:

$$\rho(\gamma).A = A^{-1}\rho(\gamma)A \qquad A \in \mathrm{PSL}_2(\mathbb{R}),\ \gamma \in \pi_1(S_g).$$

Note that $\mathcal{DF}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$ is preserved by this action; let $\mathrm{Fuchs}(S_g)$ denote the quotient of $\mathcal{DF}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$ by this action. The following theorem is essentially a fancy version of the classical uniformization theorem.

Theorem 3.1. *There is a homeomorphism* $\mathrm{Fuchs}(S_g) \cong \mathrm{Teich}(S_g)$*.*

Partial proof. We will only define the map $\mathrm{Teich}(S_g) \to \mathrm{Fuchs}(S_g)$ that turns out to be a homeomorphism. Given $[\varphi : S_g \to X] \in \mathrm{Teich}(S_g)$, choose a universal covering map $\pi : \widetilde{S}_g \to S_g$ and a locally biholomorphic universal covering map $\pi' : \mathbb{H}^2 \to X$ (the existence of π' is the content of the uniformization theorem). Let $\widetilde{\varphi} : \widetilde{S}_g \to \mathbb{H}^2$ be a lift of φ. For any deck transformation $\gamma : \widetilde{S}_g \to \widetilde{S}_g$ for π, the map $\widetilde{\varphi} \circ \gamma \circ \widetilde{\varphi}^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is a deck transformation for π', and hence is an element of $\mathrm{PSL}_2(\mathbb{R})$. Now define

$$\begin{aligned} f : \mathrm{Teich}(S_g) &\to \mathrm{Fuchs}(S_g) \\ [\varphi] &\mapsto [\gamma \mapsto \widetilde{\varphi} \circ \gamma \circ \widetilde{\varphi}^{-1}]. \end{aligned}$$

The map f does not depend on any of the choices we have made, and is the desired homeomorphism. □

Since $\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{PSL}_2(\mathbb{C})$, we can make the following definition.

Definition 3.2. A representation $\eta : \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$ is *quasi-Fuchsian* if there exists a quasiconformal map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ and a Fuchsian representation $\rho : \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{R})$ such that

$$\eta(\gamma) = f \circ \rho(\gamma) \circ f^{-1}, \qquad \forall \gamma \in \pi_1(S_g).$$

The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{C}))$ by conjugation just as above, and the space of all quasi-Fuchsian representations of $\pi_1(S_g)$ is naturally a subspace of this Hom space, just as above. Denote by $\mathrm{QFuchs}(S_g)$ the quotient of the space of quasi-Fuchsian representations of $\pi_1(S_g)$ by this action. △

Lemma 3.3. *Let* $A \in \mathrm{PSL}_2(\mathbb{C})$ *and* $\mu \in \mathrm{Bel}(\mathbb{CP}^1)$*. If* $A^*\mu = \mu$*, then for every quasiconformal* $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ *satisfying the Beltrami equation for* μ*, we have* $f \circ A \circ f^{-1} \in \mathrm{PSL}_2(\mathbb{C})$*.*

Proof. For $0 \in \mathrm{Bel}(\mathbb{CP}^1)$, we have

$$(f \circ A \circ f^{-1})^* 0 = (f^{-1})^* A^* f^* 0 = (f^{-1})^* A^* \mu = (f^{-1})^* \mu = 0.$$

Therefore $f \circ A \circ f^{-1}$ is a holomorphic homeomorphism $\mathbb{CP}^1 \to \mathbb{CP}^1$. □

Similarly to Theorem 3.1, we have

Theorem 3.4 (Bers' Simultaneous Uniformization)**.** *There is a homeomorphism* $\mathrm{QFuchs}(S_g) \cong \mathrm{Teich}(S_g) \times \mathrm{Teich}(\overline{S_g})$*.*

Recall that $\overline{S_g}$ denotes the same underlying smooth surface as S_g, endowed with the opposite orientation. Of course, there is only one equivalence class of genus g surfaces up to orientation-preserving diffeomorphism, but we will find it more convenient to use $\overline{S_g}$ in some places rather than S_g. Similarly, given a Riemann surface X, we denote by $\overline{X}$ the underlying topological surface of X, endowed with the opposite complex structure (and hence opposite oreintation); if $J : T_{\mathbb{R}}X \to T_{\mathbb{R}}X$ defines the complex structure for X, then $-J : T_{\mathbb{R}}\overline{X} \to T_{\mathbb{R}}\overline{X}$ defines the complex structure for $\overline{X}$. We can also describe $\overline{X}$ as follows: if $\pi : \mathbb{H}^2 \to X$ is a locally biholomorphic covering with deck transformation group $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{z \mapsto \overline{z}} & \overline{\mathbb{H}}^2 \\ \big\downarrow \pi & & \big\downarrow \overline{\pi} \\ X & \xrightarrow{\mathrm{Id}} & \overline{X} \end{array}$$

where $\mathrm{Id} : X \to \overline{X}$ is the identity map on the underlying topological surface, and $\overline{\pi} : \overline{\mathbb{H}}^2 \to \overline{X}$ is the quotient of $\overline{\mathbb{H}}^2$ by Γ.

Fix some orientation-preserving diffeomorphism $\varphi_0 : S_g \to X$ from S_g to a Riemann surface X, and fix a locally biholomorphic covering $\pi : \mathbb{H}^2 \to X$. The map φ_0 induces an identification $\varphi_0^* : \mathrm{Bel}(X) \xrightarrow{\sim} \mathrm{Bel}(S_g)$. Any $\mu \in \mathrm{Bel}(X)$ lifts to some $\widetilde{\mu} := \pi^*\mu \in \mathrm{Bel}(\mathbb{H}^2)$ that is invariant under the deck transformation group $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$; given any deck transformation $\gamma \in \Gamma$, we have $\gamma^*\widetilde{\mu} = (\pi \circ \gamma)^*\mu = \pi^*\mu = \widetilde{\mu}$. The map φ_0 is also an orientation-preserving diffeomorphism $\overline{S_g} \to \overline{X}$, and hence also induces an identification $\varphi_0^* : \mathrm{Bel}(\overline{X}) \xrightarrow{\sim} \mathrm{Bel}(\overline{S_g})$. Letting $\overline{\pi}$ be as in the above diagram, we can also lift any $\nu \in \mathrm{Bel}(\overline{X})$ to some $\widetilde{\nu} := \overline{\pi}^*\nu \in \mathrm{Bel}(\overline{\mathbb{H}}^2)$ that is invariant under Γ. We can therefore define $(\widetilde{\mu}, \widetilde{\nu}) \in \mathrm{Bel}(\mathbb{CP}^1)$ by

$$(\widetilde{\mu}, \widetilde{\nu})_z := \begin{cases} \widetilde{\mu}_z & \text{if } z \in \mathbb{H}^2 \\ \widetilde{\nu}_z & \text{if } z \in \overline{\mathbb{H}}^2. \end{cases}$$

Note that this definition depends on the choice of φ_0 and the choice of π.

Definition 3.5. Fix some orientation-preserving diffeomorphism $\varphi_0 : S_g \to X$ from S_g to a Riemann surface X, and fix a universal covering map $\pi : \mathbb{H}^2 \to X$. Given $\mu \in \mathrm{Bel}(S_g)$, $\nu \in \mathrm{Bel}(\overline{S_g})$, let $f_{\mu,\nu} : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the unique quasiconformal map with Beltrami differential $(\widetilde{\varphi_0^*\mu}, \widetilde{\varphi_0^*\nu})$ that fixes 0, 1, and ∞. Note[1] that φ_0 and π induce an identification $\pi_1(S_g) \cong \pi_1(X) \cong \mathrm{Deck}(\pi) \subseteq \mathrm{PSL}_2(\mathbb{R})$. We define the quasi-Fuchsian representation $\rho_{\mu,\nu}$ by

$$\rho_{\mu,\nu}(\gamma) := f_{\mu,\nu} \circ \gamma \circ f_{\mu,\nu}^{-1}, \qquad \forall \gamma \in \pi_1(S_g).$$

By Lemma 3.3, $\rho_{\mu,\nu}$ is a well-defined quasi-Fuchsian representation.

△

[1] Actually, different choices of basepoints in S_g, X, and $\mathbb{H}^2$ will induce different identifications $\pi_1(S_g) \cong \mathrm{Deck}(\pi)$. However, all of these identifications differ only by postcomposition by an inner automorphism of $\mathrm{PSL}_2(\mathbb{C})$, and so no matter which basepoints we choose, $\rho_{\mu,\nu}$ will define the same point of $\mathrm{QFuchs}(S_g)$.

Partial proof of Theorem 3.4. Let $\Phi_{S_g} : \mathrm{Bel}(S_g) \to \mathrm{Teich}(S_g)$ and $\Phi_{\overline{S_g}} : \mathrm{Bel}(\overline{S_g}) \to \mathrm{Teich}(\overline{S_g})$ be the quotient maps. Fix some orientation-preserving diffeomorphism $\varphi_0 : S_g \to X$ from S_g to a Riemann surface X, and fix a universal covering map $\pi : \mathbb{H}^2 \to X$, and define the map

$$\begin{aligned} \mathcal{QF} : \mathrm{Bel}(S_g) \times \mathrm{Bel}(\overline{S_g}) &\to \mathrm{QFuchs}(S_g) \\ (\mu, \nu) &\mapsto [\rho_{\mu,\nu}]. \end{aligned}$$

The proof is completed by showing that $\mathcal{QF}$ factors through $\Phi_{S_g} \times \Phi_{\overline{S_g}}$, and that the induced map $QF : \mathrm{Teich}(S_g) \times \mathrm{Teich}(\overline{S_g})$ is a homeomorphism. For a proof of part of this, see [Hub06]. □

Notice that we have a "diagonal" inclusion, which can be written as either

$$\begin{aligned} \mathrm{Teich}(S_g) &\hookrightarrow \mathrm{Teich}(S_g) \times \mathrm{Teich}(\overline{S_g}) \\ \tau &\mapsto (\tau, \overline{\tau}) \end{aligned}$$

or

$$\begin{aligned} \mathrm{Fuchs}(S_g) &\hookrightarrow \mathrm{QFuchs}(S_g) \\ \rho &\mapsto \rho, \end{aligned}$$

where $\overline{\tau}$ denotes, for $\tau = [\varphi : S_g \to X] \in \mathrm{Teich}(S_g)$, the element

$$[\overline{S_g} \xrightarrow{\mathrm{Id}} S_g \xrightarrow{\varphi} X \xrightarrow{\mathrm{Id}} \overline{X}] \in \mathrm{Teich}(\overline{S_g}).$$

Equivalently, for $\tau = [\mu]$, we have $\overline{\tau} = [\overline{\mu}]$.

Given $(\mu, \nu) \in \mathrm{Bel}(S_g) \times \mathrm{Bel}(\overline{S_g})$, the universal covering

$$f_{\mu,\nu}(\mathbb{H}^2) \to f_{\mu,\nu}(\mathbb{H}^2)/\rho_{\mu,\nu}(\pi_1(S_g))$$

defines a projective structure on the Riemann surface $X = f_{\mu,\nu}(\mathbb{H}^2)/\rho_{\mu,\nu}(\pi_1(S_g))$. This projective structure depends only on the point $(\tau, \kappa) = \Phi_{S_g} \times \Phi_{\overline{S_g}}(\mu, \nu) \in \mathrm{Teich}(S_g) \times \mathrm{Teich}(\overline{S_g})$ (see [Hub06]), and so we denote it by $\sigma_\tau(\tau, \kappa) \in \pi^{-1}(\tau) \subseteq \mathrm{MProj}(S_g)$. Similarly, we define $\sigma_\kappa(\tau, \kappa) \in \mathrm{MProj}(\overline{S_g})$ via the universal covering $f_{\mu,\nu}(\overline{\mathbb{H}}^2) \to f_{\mu,\nu}(\overline{\mathbb{H}}^2)/\rho_{\mu,\nu}(\pi_1(\overline{S_g}))$. Finally, observe that $\sigma_{\mathrm{Fuchs}}(\tau) = \sigma_\tau(\tau, \overline{\tau})$.

3.2 The Bers maps

Recall that, by Lemma 2.16, the difference between two projective structures in $\pi^{-1}([\varphi : S_g \to X])$ can be seen as an element of $\mathrm{QD}(X)$.

Definition 3.6 (Bers projection and Bers embedding)**.** Let S be a closed, oriented surface. Let $\Phi_S : \mathrm{Bel}(S) \to \mathrm{Teich}(S)$ be the quotient map, and let $\kappa = [\psi : \overline{S} \to Y] \in \mathrm{Teich}(\overline{S})$. We define the *Bers projection* (based at κ)

$$\begin{aligned} \widetilde{\Psi}_\kappa : \mathrm{Bel}(S) &\to \mathrm{QD}(Y) \\ [\varphi, \mu] &\mapsto \sigma_\kappa(\Phi_S([\varphi, \mu]), \kappa) - \sigma_\kappa(\overline{\kappa}, \kappa). \end{aligned}$$

Clearly $\widetilde{\Psi}_\kappa$ factors through $\Phi_S : \mathrm{Bel}(S) \to \mathrm{Teich}(S)$, and hence descends to a map $\Psi_\kappa : \mathrm{Teich}(S) \to \mathrm{QD}(Y)$, which we call the *Bers embedding* (based at κ). As shorthand we will call these maps the *Bers maps.* △

Analyticity of $\widetilde{\Psi}_\kappa$ is fairly straightforward, and we also have the following lemma.

Lemma 3.7. *The Bers embedding is injective.*

Proof. See propositions 9.8-9.10 of [Wri]. □

The following lemma is crucial for Theorem 3.9, and can be seen as the point at which the stories of §1 and §2 intersect.

Lemma 3.8 (Ahlfors-Weill)**.** *Let* $B_{1/2}(0) \subseteq \mathrm{QD}(Y)$ *be the ball of radius* $1/2$ *about* 0 *in* $\mathrm{QD}(Y)$*. This ball lies in the image of* $\widetilde{\Psi}_\kappa$*, and there exists a holomorphic section* $\sigma : B_{1/2}(0) \to \mathrm{Bel}(S)$ *of* $\widetilde{\Psi}_\kappa$*.*

Proof. See Theorem 6.3.10 of [Hub06]. □

Theorem 3.9 (Teichmüller spaces are complex manifolds)**.** *About each* $\tau \in \mathrm{Teich}(S)$ *there is an open set* $U_\tau \subseteq \mathrm{Teich}(S)$ *such that* $\{(U_\tau, \Psi_{\overline{\tau}})\}_{\tau \in \mathrm{Teich}(S)}$ *is a system of coordinate charts endowing* $\mathrm{Teich}(S)$ *with the structure of a complex* $(3g-3)$*-dimensional manifold.*

Proof. See pages 265-266 of [Hub06]. □

As a consequence of Theorem 3.9, we have

Theorem 3.10. *Given* $\tau = [\varphi : S \to X] \in \mathrm{Teich}(S)$*, the holomorphic cotangent space* $T^*_\tau \mathrm{Teich}(S)$ *is canonically isomorphic to* $\mathrm{QD}(X)$*, and the holomorphic tangent space* $T_\tau \mathrm{Teich}(S)$ *is canonically isomorphic to* $\mathrm{bel}(S)/\mathrm{bel}_0(S)$*, where* $\mathrm{bel}(S)$ *is the tangent space at* 0 *to* $\mathrm{Bel}(S)$*, and* $\mathrm{bel}_0(S) \subseteq \mathrm{bel}(S)$ *is the linear subspace of differentials* $\mu \in \mathrm{bel}(S)$ *satisfying* $\int \mu q = 0$ *for every* $q \in \mathrm{QD}(X)$*.*

3.3 Differentiating the Bers maps

(Forthcoming)

A Notation for Complex Manifolds

A.1 Definitions for general complex manifolds

Let M be a complex manifold of real dimension $2n$. Any coordinates we use are assumed to be given by some coordinate chart in the atlas that determines the complex structure on M. Throughout these notes, if V is a vector space/bundle, we denote by V^* its linear dual, *not* its complex conjugate.

- Given any vector bundle $E \to M$, we denote by $\Gamma(E)$ the space of smooth sections of E. Unfortunately, we also often use the letter "Γ" to denote a Fuchsian group, although it should always be clear in each context which use of the letter is meant.

- $T_{\mathbb{R}}M$ is the tangent bundle to M as a real $2n$-manifold. It is a real rank $2n$ vector bundle with an $\mathbb{R}$-basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ at every point. It has a natural almost-complex structure given by

$$J_M : \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$$
$$\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$$

  at every point. Therefore the fibers of $T_{\mathbb{R}}M$ admit the structure of an n-dimensional complex vector space with $\mathbb{C}$-basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, where the complex scalar multiplication is given by $(a+bi)v := av + bJ_X(v)$. In this way, $T_{\mathbb{R}}M$ can also be viewed as a complex rank n vector bundle.

- $T^*_{\mathbb{R}}M = (T_{\mathbb{R}}M)^*$ is the cotangent bundle to M as a real $2n$-manifold. It is a real rank $2n$ vector bundle with an $\mathbb{R}$-basis $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ at every point. It has a natural almost-complex structure given by

$$J^*_M : dx_i \mapsto dx_i \circ J_M = -dy_i$$
$$dy_i \mapsto dy_i \circ J_M = dx_i$$

  at every point. Therefore the fibers of $T^*_{\mathbb{R}}M$ admit the structure of an n-dimensional complex vector space with $\mathbb{C}$-basis $dx_1, \ldots, dx_n$, where the complex scalar multiplication is given by $(a+bi)\varphi := a\varphi + bJ^*_M\varphi$. In this way, $T^*_{\mathbb{R}}M$ can also be viewed as a complex rank n vector bundle. Note that it is the complex dual bundle to the complex vector bundle $T_{\mathbb{R}}M$.

- $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified tangent bundle to M. It is a complex rank $2n$ vector bundle with a $\mathbb{C}$-basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ at every point.

- $T^*_{\mathbb{C}}M = (T_{\mathbb{C}}M)^* = T^*_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified cotangent bundle to M. It is a complex rank $2n$ vector bundle with a $\mathbb{C}$-basis $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ at every point.

- TM is the holomorphic tangent bundle to M. It is a complex rank n sub-vector bundle of $T_{\mathbb{C}}M$ with a $\mathbb{C}$-basis $\frac{\partial}{\partial z_1} := \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}), \ldots, \frac{\partial}{\partial z_n} := \frac{1}{2}(\frac{\partial}{\partial x_n} - i\frac{\partial}{\partial y_n})$ at every point. The fiber of TM at a point is the eigenspace of i for the complex-linear extension $J_M \otimes_{\mathbb{R}} \mathbb{C}$ of J_M. As a complex vector bundle, TM is isomorphic to $T_{\mathbb{R}}M$ via the isomorphism that at every fiber takes $\frac{\partial}{\partial z_i}$ to $\frac{\partial}{\partial x_i}$.

- $\overline{TM}$ is the anti-holomorphic tangent bundle to M. It is a complex rank n sub-vector bundle of $T_{\mathbb{C}}M$ with a $\mathbb{C}$-basis $\frac{\partial}{\partial \overline{z_1}} := \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}), \dots, \frac{\partial}{\partial \overline{z_n}} := \frac{1}{2}(\frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n})$ at every point. The fiber of $\overline{TM}$ at a point is the eigenspace of $-i$ for the complex-linear extension $J_M \otimes_{\mathbb{R}} \mathbb{C}$ of J_M.

- T^*M is the holomorphic cotangent bundle to M. It is a complex rank n sub-vector bundle of $T^*_{\mathbb{C}}M$ with a $\mathbb{C}$-basis $dz_1 := dx_1 + idy_1, \dots, dz_n := dx_n + idy_n$ at every point. The fiber of T^*M at a point is the eigenspace of i for the complex-linear extension $J^*_M \otimes_{\mathbb{R}} \mathbb{C}$ of J^*_M. As a complex vector bundle, T^*M is isomorphic to $T^*_{\mathbb{R}}M$ via the isomorphism that at every fiber takes dz_i to dx_i.

- $\overline{T^*M}$ is the anti-holomorphic cotangent bundle to M. It is a complex rank n sub-vector bundle of $T^*_{\mathbb{C}}M$ with a $\mathbb{C}$-basis $d\overline{z_1} := dx_1 - idy_1, \dots, d\overline{z_n} := dx_n - idy_n$ at every point. The fiber of $\overline{T^*M}$ at a point is the eigenspace of $-i$ for the complex-linear extension $J^*_M \otimes_{\mathbb{R}} \mathbb{C}$ of J^*_M.

- $\bigwedge^k T^*_{\mathbb{R}}M$ is the kth alternating power of $T^*_{\mathbb{R}}M$. It is a real rank $\binom{2n}{k}$ quotient vector bundle of $(T^*_{\mathbb{R}}M)^{\otimes k}$ with an $\mathbb{R}$-basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}\}_{i_1 < \cdots < i_k}$ at every point. Smooth sections of $\bigwedge^k T^*_{\mathbb{R}}M$ are called *real differential k-forms*, and nowhere-vanishing smooth sections of $\bigwedge^{2n} T^*_{\mathbb{R}}M$ are called *volume forms*.

- $\mathrm{Sym}^k(T^*_{\mathbb{R}}M)$ is the kth symmetric power of $T^*_{\mathbb{R}}M$. It is a real rank $\binom{n+k-1}{k}$ quotient vector bundle of $(T^*_{\mathbb{R}}M)^{\otimes k}$ with an $\mathbb{R}$-basis

$$\{dx_1^{i_1} \cdots dx_n^{i_n}\}_{i_1 + \cdots + i_n = k}$$

  at every point. When $k = 2$, for $p \in M$ and $v, w \in (T_{\mathbb{R}}M)_p$, we have $dadb(v, w) = \frac{1}{2}(da(v)db(w) + da(w)db(v))$ for a, b among the x_i, y_j. A section g of $\mathrm{Sym}^2(T^*_{\mathbb{R}}M)$ is called a *Riemannian metric* if the bilinear form $g_p(\cdot, \cdot)$ is positive definite at every point $p \in M$.

- $\bigwedge^k T^*_{\mathbb{C}}M$ is the kth alternating power of $T^*_{\mathbb{C}}M$. It is a complex rank $\binom{2n}{k}$ quotient vector bundle of $(T^*_{\mathbb{C}}M)^{\otimes k}$ with an $\mathbb{C}$-basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}\}_{i_1 < \cdots < i_k}$ at every point.

- $\mathrm{Sym}^k(T^*_{\mathbb{C}}M)$ is the kth symmetric power of $T^*_{\mathbb{C}}M$. It is a complex rank $\binom{n+k-1}{k}$ quotient vector bundle of $(T^*_{\mathbb{C}}M)^{\otimes k}$ with a $\mathbb{C}$-basis

$$\{dx_1^{i_1} \cdots dx_n^{i_n}\}_{i_1 + \cdots + i_n = k}$$

  at every point.

A.2 Definitions for Riemann surfaces

Let X be a complex manifold of dimension 1, called a *Riemann surface*.

- The *canonical bundle* of X is T^*X, sometimes denoted K, and has $\mathbb{C}$-basis dz at every point. Holomorphic sections $\omega \in \Gamma(K)$ are called *abelian differentials*. Note that $K^* = TX$, with $\mathbb{C}$-basis $\frac{\partial}{\partial z}$ at every point.

- $K^2 \coloneqq K \otimes_{\mathbb{C}} K = \mathrm{Sym}^2(K) \subset \mathrm{Sym}^2(T^*_{\mathbb{C}}X)$. Holomorphic sections $q \in \Gamma(K^2)$ are called *holomorphic quadratic differentials*.

- We have an inclusion $\bigwedge^2 T^*_{\mathbb{C}}X \hookrightarrow (T^*_{\mathbb{C}}X)^{\otimes 2}$ given by $\alpha \wedge \beta \mapsto \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)$. We also have an inclusion $\mathrm{Sym}^2(T^*_{\mathbb{C}}X) \hookrightarrow (T^*_{\mathbb{C}}X)^{\otimes 2}$ given by $\alpha\beta \mapsto \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. Then

$$\begin{aligned} dz \otimes d\bar{z} &= (dx + idy) \otimes (dx - idy) \\ &= \underbrace{dx^2 + dy^2}_{\text{Riemannian metric}} \quad -2i \underbrace{dx \wedge dy}_{\text{Volume form}} \\ &= dzd\bar{z} - dz \wedge d\bar{z}. \end{aligned}$$

B Remarks on the definition of $\mathrm{Mod}(S)$

When the genus of X is at least 2, there is an even stronger formulation of Theorem 1.17.

Theorem B.1. *If S and S' are closed surfaces of genus at least 2, then any homotopy equivalence $S \to S'$ is homotopic to a diffeomorphism.*

Proof. See Section 8.3.2 of [FM12]. ☐

One often sees the mapping class group defined in terms of diffeomorphisms, rather than in terms of quasiconformal maps. The following lemma and its corollary tell us that these definitions are equivalent. Furthermore, Theorem B.5 tells us the same for homeomorphisms.

Lemma B.2. *Every quasiconformal map $\varphi : S_g \to X$ is homotopic to an orientation-preserving diffeomorphism $\psi : S_g \to X$.*

Proof. For $g \geq 2$, the claim follows from Theorem B.1. For $g = 0$, the claim follows from the fact that $\pi_2(S_0) = \mathbb{Z}$. For $g = 1$, it follows from Theorem 2.5 of Farb-Margalit that every homeomorphism $S_1 \to S_1$ is homotopic to a special-linear map with integral coefficients, which is clearly an orientation-preserving diffeomorphism. ☐

Corollary B.3. *Let $\mathrm{Diff}^+(S_g)$ denote the group of orientation-preserving diffeomorphisms $S_g \to S_g$, and let $\mathrm{Diff}^+_0(S_g)$ denote the subgroup of $\mathrm{Diff}^+(S_g)$ of diffeomorphisms that are homotopic to the identity map. Then the inclusion $\mathrm{Diff}^+(S_g) \hookrightarrow \mathrm{QC}(S_g)$ induces an isomorphism*

$$\mathrm{Diff}^+(S_g)/\mathrm{Diff}^+_0(S_g) \xrightarrow{\sim} \mathrm{QC}(S_g)/\,\mathrm{QC}_0(S_g) = \mathrm{Mod}(S_g).$$

Let us return to the case where $g \geq 2$. With respect to the compact-open topology, $\mathrm{Diff}_0^+(S_g)$ is the connected component of the identity in $\mathrm{Diff}^+(S_g)$, so that $\mathrm{Diff}^+(S_g)/\mathrm{Diff}_0^+(S_g) = \pi_0\mathrm{Diff}^+(S_g)$. One way of seeing this is as a consequence of the following two theorems. Let $\mathrm{Homeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms S_g with the compact-open topology.

Theorem B.4 (Theorem 1.12 of [FM12])**.** *Let $\varphi, \psi \in \mathrm{Homeo}^+(S_g)$ be homotopic functions. Then φ and ψ lie in the same connected component of $\mathrm{Homeo}^+(S_g)$.*

Theorem B.5 (Theorem 1.2 of [Bol09])**.** *The inclusion $\mathrm{Diff}^+(S_g) \hookrightarrow \mathrm{Homeo}^+(S_g)$ induces an isomorphism $\pi_0\mathrm{Diff}^+(S_g) \xrightarrow{\sim} \pi_0\mathrm{Homeo}^+(S_g)$.*

Corollary B.6. *Let $\varphi, \psi \in \mathrm{Diff}^+(S_g)$ be homotopic functions. Then φ and ψ lie in the same connected component of $\mathrm{Diff}^+(S_g)$.*

Proof. By Theorem B.4, φ and ψ lie in the same connected component of $\mathrm{Homeo}^+(S_g)$. By Theorem B.5, φ and ψ therefore also lie in the same connected component of $\mathrm{Diff}^+(S_g)$. □

I am not aware of any analogue to Corollary B.6 with $\mathrm{Diff}^+(S_g)$ replaced with $\mathrm{QC}(S_g)$, and I don't know whether or not this is an open problem.

C Computing the dimension of $\mathrm{QD}(X)$ with Riemann-Roch

Let S be a closed oriented surface. Let Ω_X denote the sheaf of holomorphic 1-forms on a Riemann surface X. Then we have the following theorem, which is a consequence of Bers' embedding theorem.

Theorem C.1. *The holomorphic cotangent space to $\mathrm{Teich}(S)$ at the point $[\varphi : S \to X]$ is isomorphic to $H^0(X, \Omega_X^{\otimes 2})$.*

We wish to compute the dimension of $\mathrm{QD}(X) = H^0(X, \Omega_X^{\otimes 2})$. This will be done by appealing to Serre duality and the Riemann-Roch theorem. We will also give a computation by appealing to Theorem C.1 and the Fenchel-Nielsen coordinates on $\mathrm{Teich}(S)$.

Theorem C.2. *When $g \geq 2$, $\dim_{\mathbb{C}} H^0(X, \Omega_X^{\otimes 2}) = 3g - 3$.*

We will see that Serre duality also gives the following theorem.

Theorem C.3. *The holomorphic tangent space to $\mathrm{Teich}(S)$ at the point $[\varphi : S \to X]$ is isomorphic to $H^1(X, \Omega_X^{\otimes -1})$.*

Note that $\Omega_X^{\otimes -1}$ is the sheaf of holomorphic vector fields on X. Note also that $H^1(X, \Omega_X^{\otimes -1})$ is the space of all infinitesimal deformations of the complex structure on X in the sense of Kodaira-Spencer. The Bers embedding identifies the tangent space to $\mathrm{Teich}(S)$ at the point $[\varphi : S \to X]$ with a quotient of the vector space bel(X) of L^∞ global sections of $\overline{\Omega_X} \otimes {\Omega_X}^{\otimes -1}$, where $\overline{\Omega_X}$ is the space of antiholomorphic 1-forms on X. It is nontrivial, but not too difficult, to show directly that these two descriptions of the tangent space to Teich(S) are equivalent.

We will also concern ourselves with the space $H^0(X, \Omega_X)$ of global holomorphic 1-forms on X. These forms are of course of natural interest, and also admit a very concrete interpretation in terms of *translation surfaces*, which are of particular contemporary interest. Serre duality will give the following theorem.

Theorem C.4. $\dim_{\mathbb{C}} H^0(X, \Omega_X) = g$.

This is a particular instance of Hodge symmetry, which indeed also follows from Serre duality.

C.1 Serre Duality and Riemann-Roch

In this section we quote analytic versions of Serre duality and the Riemann-Roch theorem. Denote by $\mathcal{O}(-)$ the sheaf of sections of a vector bundle, and denote by $c_1(-)$ the first Chern class of a line bundle.

Theorem C.5 (Serre duality, Theorem A9.14 of [Hub06])**.** *Let V be a holomorphic vector bundle on a compact complex manifold X of dimension n, and let V^* be the dual vector bundle. Then*

$$H^k(X, \mathcal{O}(V)) \textit{ is dual to } H^{n-k}(X, \mathcal{O}(V^*) \otimes \Omega_X^{\otimes n}).$$

In particular, we will apply Serre duality in the following case. Let d be an integer, and let X be a compact Riemann surface. Then

$$H^1(X, \Omega_X^{\otimes d}) \text{ is dual to } H^0(X, \Omega_X^{\otimes (1-d)}). \tag{3}$$

Observe that Theorem C.3 now follows immediately from (3) and Theorem C.1.

Theorem C.6 (Riemann-Roch, Theorem A10.0.1 of [Hub06])**.** *Let L be a holomorphic line bundle on a compact Riemann surface X of genus g. Then*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}(L)) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}(L)) = c_1(L) + 1 - g.$$

C.2 Dimension Counts

In order to apply the Riemann-Roch theorem, we must know how to compute the relevant Chern classes. This computation is made by appealing to the following two lemmas.

Lemma C.7 (Proposition A10.2.1 of [Hub06])**.** *On a compact Riemann surface X, let L be a holomorphic line bundle and let D be a Weil divisor such that $\mathcal{O}(L) = \mathcal{O}(D)$. Then $c_1(L) = \deg(D)$.*

Lemma C.8 (Proposition 1.14 of Chapter V of [Mir95])**.** *Let X be a compact Riemann surface. Then X has a canonical divisor of degree $2g - 2$.*

It follows from the above lemmas that

$$c_1(\Omega_X^{\otimes d}) = d(2g-2). \tag{4}$$

We can now prove Theorem C.2.

First proof of Theorem C.2. This proof appears as the proof of Proposition A10.3.2 in [Hub06]. Since $c_1(\Omega_X^{\otimes -1}) = 2-2g < 0$, we have $H^0(X, \Omega_X^{\otimes -1}) = 0$. By Serre duality, $H^1(X, \Omega_X^{\otimes 2}) = 0$. Then Riemann-Roch gives

$$\begin{aligned} \dim_{\mathbb{C}} H^0(X, \Omega_X^{\otimes 2}) - \dim_{\mathbb{C}} H^1(X, \Omega_X^{\otimes 2}) &= c_1(\Omega_X^{\otimes 2}) + 1 - g \\ \dim_{\mathbb{C}} H^0(X, \Omega_X^{\otimes 2}) - 0 &= 2(2g-2) + 1 - g \qquad \text{by (4)} \\ \dim_{\mathbb{C}} H^0(X, \Omega_X^{\otimes 2}) &= 3g-3. \end{aligned}$$

□

We can also prove Theorem C.2 be appealing to the Fenchel-Nielsen coordinates on Teich(S).

Theorem C.9 (Fenchel-Nielsen coordinates)**.** *Let S be a closed topological surface of genus $g \geq 2$. Then* Teich(S) *is diffeomorphic to* $\mathbb{R}^{6g-6}$.

Second proof of Theorem C.2. By Theorem C.9, the holomorphic cotangent space to Teich(S) at any point $[\varphi : S \to X]$ has real dimension $6g-6$, and hence complex dimension $3g-3$. By Theorem C.1, this space is isomorphic to $H^0(X, \Omega_X^{\otimes 2})$. □

We also remark that a heuristic dimension count of $\dim_{\mathbb{C}} H^0(X, \Omega_X^{\otimes 2})$ can be made by appealing to the fact that global sections of $\Omega_X^{\otimes 2}$ are in one-to-one correspondence with planar polygons whose sides come in parallel pairs.

We can also now prove Theorem C.4.

Proof of Theorem C.4. This proof appears as the proof of Proposition A10.1.1 in [Hub06]. The short exact sequence of sheaves

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_X \to \Omega_X \to 0$$

induces a long exact sequence in cohomology. Part of this long exact sequence is the short exact sequence

$$0 \to H^0(X, \Omega_X) \to H^1(X, \underline{\mathbb{C}}) \to H^1(X, \mathcal{O}_X) \to 0.$$

Serre duality gives that the left- and right-hand terms are dual, and hence

$$\dim_{\mathbb{C}} H^0(X, \Omega_X) = \frac{1}{2} \dim_{\mathbb{C}} H^1(X, \underline{\mathbb{C}}) = g.$$

□

D A perspective on the Weil-Petersson metric

In this section we construct the Weil-Petersson metric. Let $X \cong \mathbb{H}^2/\Gamma$ be a closed Riemann suface, and let $\mathrm{Teich}(X)$ be its Teichmüller space. We will present the Weil-Petersson metric on $\mathrm{Teich}(X)$ as an explicit isomorphism $T^* \mathrm{Teich}(X) \xrightarrow{\sim} T \mathrm{Teich}(X)$.

Note throughout that all the bundle isomorphisms we consider below are constructions that work for any complex 1-manifold; they don't just work because $\mathbb{H}^2$ is contractible, so that all C^∞-bundles of the same dimension are vacuously isomorphic! In particular, these constructions descend naturally to any quotient manifold of $\mathbb{H}^2$. We're really only working in $\mathbb{H}^2$ because we want to write down explicit coordinate representations of forms. Thus, we're using the upper half-plane model of $\mathbb{H}^2$.

Consider the hyperbolic metric $ds^2 = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy) \in \Gamma((T^*_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2})$. This is an everywhere nonvanishing section of $(T^*_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2}$, and hence spans a line subbundle $L \hookrightarrow (T^*_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2}$. Furthermore, ds^2 is a Riemannian metric, and hence determines an isomorphism

$$\begin{gathered} T_{\mathbb{R}}\mathbb{H}^2 \leftrightarrow T^*_{\mathbb{R}}\mathbb{H}^2 \\ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \leftrightarrow \frac{1}{y^2}dx, \frac{1}{y^2}dy, \end{gathered}$$

and hence an isomorphism

$$\begin{gathered} (T_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2} \leftrightarrow (T^*_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2} \\ \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \leftrightarrow \frac{1}{y^4} dx_i \otimes dx_j, \end{gathered}$$

where we write $x_1 = x$ and $x_2 = y$ for simplicity. Under this isomorphism, we have

$$ds^2 = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy) \leftrightarrow \frac{1}{y^6}\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}\right)$$

The section $\frac{1}{y^6}\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}\right) \in \Gamma((T_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2})$ is also everywhere nonvanishing, and hence spans a line subbundle $L' \hookrightarrow (T_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2}$.

Now consider the line bundle $L^* = \mathrm{Hom}_{\mathbb{R}}(L, \mathbb{H}^2 \times \mathbb{R})$, and denote by $\frac{1}{ds^2} \in \Gamma(L^*)$ the form satisfying $\frac{1}{ds^2}(ds^2) = 1$ at every point of $\mathbb{H}^2$. We have an isomorphism $L^* \xrightarrow{\sim} L'$ taking $\frac{1}{ds^2}$ to $\frac{y^2}{2}\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}\right)$, which we shall also denote by $\frac{1}{ds^2}$. This realizes $\frac{1}{ds^2}$ as a section of $(T_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2}$ satisfying $\frac{1}{ds^2}(ds^2) = 1$ at every point of $\mathbb{H}^2$. Under the inclusion $(T_{\mathbb{R}}\mathbb{H}^2)^{\otimes 2} \hookrightarrow (T_{\mathbb{C}}\mathbb{H}^2)^{\otimes 2}$, we write $\frac{1}{ds^2}$ in complex notation:

$$\frac{1}{ds^2} = \mathrm{Im}(z)^2 \left(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}} \otimes \frac{\partial}{\partial z}\right).$$

We now define the Weil-Petersson Riemannian metric on $\mathrm{Teich}(X)$. We define the metric via its induced isomorphism $\mathrm{QD}(X) = T^* \mathrm{Teich}(X) \xrightarrow{\sim} T \mathrm{Teich}(X) =$

$\mathrm{bel}(X)/\mathrm{bel}_0(X)$. Given $\varphi \in \mathrm{QD}(X)$, let us abuse notation and also write $\varphi = f dz \otimes dz \in \mathrm{QD}(\mathbb{H}^2)$ for the lift of φ to the universal cover $\mathbb{H}^2$ of X. Then consider

$$\overline{\varphi} \otimes \frac{1}{ds^2} = \overline{f}\mathrm{Im}(z)^2 \left(d\overline{z} \otimes d\overline{z} \otimes \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}} + d\overline{z} \otimes d\overline{z} \otimes \frac{\partial}{\partial \overline{z}} \otimes \frac{\partial}{\partial z} \right).$$

Now, notice that the line subbundle $H \hookrightarrow (T^*_{\mathbb{C}}\mathbb{H}^2)^{\otimes 2}$ spanned by $\overline{\varphi} \otimes \frac{1}{ds^2}$ is isomorphic to $\overline{T^*\mathbb{H}^2} \otimes T\mathbb{H}^2$ via the isomorphism $(d\overline{z} \otimes d\overline{z} \otimes \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}} + d\overline{z} \otimes d\overline{z} \otimes \frac{\partial}{\partial \overline{z}} \otimes \frac{\partial}{\partial z}) \mapsto 2d\overline{z} \otimes \frac{\partial}{\partial z}$. (Note: it seems that it is sometimes conventional to rather use $\cdots \mapsto d\overline{z} \otimes \frac{\partial}{\partial z}$ so that $\overline{\varphi} \otimes \frac{1}{ds^2} = \overline{f}\mathrm{Im}(z)^2 d\overline{z} \otimes \frac{\partial}{\partial z}$. This is meant to match the formal calculation $\overline{f} d\overline{z}^2 \cdot \frac{\mathrm{Im}(z)^2}{dz d\overline{z}} = \overline{f}\mathrm{Im}(z)^2 \frac{d\overline{z}^2}{dz d\overline{z}} = \overline{f}\mathrm{Im}(z)^2 \frac{d\overline{z}}{dz}$.) Under this isomorphism, we have

$$\overline{\varphi} \otimes \frac{1}{ds^2} = 2\overline{f}\mathrm{Im}(z)^2 d\overline{z} \otimes \frac{\partial}{\partial z} \in \mathrm{bel}(\mathbb{H}^2).$$

Note that $\varphi \in \mathrm{QD}(\mathbb{H}^2)$ is Γ-equivariant by construction, as is ds^2. Hence $\overline{\varphi} \otimes \frac{1}{ds^2} \in \mathrm{bel}(\mathbb{H}^2)$ is also Γ-equivariant, and hence descends to a Beltrami differential $\mu_\varphi \in \mathrm{bel}(X)$. The map $\mathrm{QD}(X) \to \mathrm{bel}(X)/\mathrm{bel}_0(X)$, $\varphi \mapsto [\mu_\varphi]$ is the isomorphism defining the Weil-Petersson metric. We remark that μ_φ is also called a *harmonic* Beltrami differential, and that the restriction of the map $\mathrm{QD}(X) \to \mathrm{bel}(X)$, $\varphi \mapsto \mu_\varphi$ to $B_{\frac{1}{2}}(0) \subseteq \mathrm{QD}(X)$ is precisely the Ahlfors-Weill section $\sigma : B_{\frac{1}{2}}(0) \to \mathrm{Bel}(X)$.

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