1 Introduction

The O’Nan-Scott Theorem gives us a classification of the subgroups $G$ of a finite symmetric group $\text{Sym}(\Omega)$. In this paper, we present the two degenerate cases where $G$ is either imprimitive or intransitive, as well as one of the cases in which $G$ is primitive. Let us briefly establish some notation. By $K \wr H$ we denote the wreath product of groups $K$ and $H$ over a set $\Gamma$. We sometimes simply write $K \wr H$ when we do not want to make reference to $\Gamma$.

The part of the O’Nan-Scott Theorem that we will prove here is as follows.

**Theorem 1** (O’Nan-Scott: Part I). Let $\Omega$ be a set with $|\Omega| = N$, and let $G < \text{Sym}(\Omega)$ with $G \not\cong A_N$. Then $G$ is a subgroup of one of the following:

(i) If $G$ is intransive, then $G \leq S_n \times S_k$ with $n + k = N$.

(ii) If $G$ is imprimitive, then $G \leq S_n \wr S_k$ with $nk = N$.

(iii) If $G$ is primitive with an abelian minimal normal subgroup, then $G \leq A\text{GL}_k(p)$ for some prime $p$ and some $k \geq 1$.

The terminology in the above theorem statement will be defined explicitly in the following sections. The proof of Theorem 1 is the content of the next two sections.

2 The Imprimitive and Intransitive Cases

2.1 The Imprimitive Case

If $G$ is imprimitive, then let $\Sigma$ be a system of blocks for $G$. The blocks all have the same size, so let $n = |\Gamma|$ for some $\Gamma \in \Sigma$, and let $k = |\Sigma|$. For each $\Gamma \in \Sigma$, let $f_\Gamma : \{1, \ldots, n\} \to \Gamma$ be some bijection. Then $S_n \wr \text{Sym}(\Sigma) \cong S_n \wr S_k$ acts on $\Omega$ by

$$(g : \Sigma \to S_n, \eta).f_\Gamma(i) = f_{\eta(\Gamma)}(g(\Gamma)(i)).$$

In words, for each block $\Gamma$, we have a copy of $S_n$ that permutes the elements of $\Gamma$, and we also have a copy of $S_k$ that permutes the blocks themselves. Since $G$ respects the system $\Sigma$ of blocks, we must have $G < S_n \wr S_k$. Clearly $nk = N$. This establishes the imprimitive case of the O’Nan-Scott Theorem. Furthermore, we have the following theorem.

**Theorem 2.** $S_n \wr S_k$ is a maximal subgroup of $\text{Sym}(\Omega)$.

Recall the following lemma of group theory.

**Lemma 3.** Every symmetric group is generated by transpositions.
Given $\alpha, \beta \in \Omega$, let us denote by $(\alpha, \beta) \in \text{Sym}(\Omega)$ the transposition of $\alpha$ and $\beta$. Also, given blocks $\Gamma, \Delta \in \Sigma$, let $(\Gamma, \Delta)$ be the element $(g, \eta)$ of $S_n \wr S_k \cong S_n \wr \text{Sym}(\Sigma) < \text{Sym}(\Omega)$, where $\eta \in \text{Sym}(\Sigma)$ is the transposition of $\Gamma$ and $\Delta$, and $g(\Gamma) = e \in S_n$ for every $\Gamma \in \Sigma$. That is to say, $(\Gamma, \Delta)$ is the “canonical” element of $S_n \wr S_k$ that transposes the blocks $\Gamma$ and $\Delta$. We now prove Theorem 2.

Proof of Theorem 2. By Lemma 3, it suffices to show that for any $\alpha, \beta \in \Omega$ and any $\varphi \in \text{Sym}(\Omega) \setminus (S_n \wr S_k)$, we have $(\alpha, \beta) \in \langle S_n \wr S_k, \varphi \rangle$. If $\alpha$ and $\beta$ are in the same block, then $(\alpha, \beta) \in S_n \wr S_k < \langle S_n \wr S_k, \varphi \rangle$, and so now suppose that we have $\alpha \in \Gamma, \beta \in \Gamma'$ for blocks $\Gamma \neq \Gamma'$. Since $\varphi \notin S_n \wr S_k$, the permutation $\varphi$ does not preserve the system of blocks. Hence there is some block $\Delta$ and $\delta, \delta' \in \Delta$ such that $\varphi(\delta) \notin T$, $\varphi(\delta') \notin T'$ for blocks $T \neq T'$. Since we have $\langle \Gamma, T \rangle, \langle \Gamma', T' \rangle \in S_n \wr S_k$, we may conjugate by these elements in order to assume without loss of generality that $\Gamma = T$ and $\Gamma' = T'$. Now, since we have $\alpha, \varphi(\delta) \in \Gamma$ and $\beta, \varphi(\delta') \in \Gamma'$, we have $(\alpha, \varphi(\delta)), (\beta, \varphi(\delta')) \in S_n \wr S_k$. We may conjugate by these transpositions in order to assume without loss of generality that $\alpha = \varphi(\delta)$ and $\beta = \varphi(\delta')$. Then, we have

$$(\alpha, \beta) = \varphi \circ (\delta, \delta') \circ \varphi^{-1} \in \langle S_n \wr S_k, \varphi \rangle,$$

as desired. We conclude that $S_n \wr S_k$ is a maximal subgroup of $\text{Sym}(\Omega)$. \hfill $\Box$

Note that if we had instead explicitly written the conjugations by $(\Gamma, T), (\Gamma', T')$, etc., then we would have

$$(\alpha, \beta) = (\Gamma, T)(\Gamma', T')(\Gamma, T)(\alpha, \varphi(\delta))(\Gamma', T')(\beta, \varphi(\delta'))\varphi(\delta, \delta')\varphi^{-1}(\Gamma', T')(\beta, \varphi(\delta'))((\Gamma, T)(\alpha, \varphi(\delta')))(\Gamma', T')(\Gamma, T).$$

2.2 The Intransitive Case

If $G$ is intransitive, then $\Omega$ is the union of at least two nontrivial orbits of $G$. Let $\Omega_1$ be one such orbit, let $\Omega_2$ be the union of the others, and let $n = |\Omega_1|, k = |\Omega_2|$. Then $G < \text{Sym}(\Omega_1) \times \text{Sym}(\Omega_2) \cong S_n \times S_k$, where $S_n \times S_k$ acts on $\Omega$ by permuting the elements of $\Omega_1$ and $\Omega_2$ independently. Clearly $n + k = N$. This establishes the intransitive case of the O’Nan-Scott Theorem. Furthermore, we have the following theorem.

Theorem 4. $S_n \times S_k$ is a maximal subgroup of $\text{Sym}(\Omega)$ unless $n = k$, in which case we have $S_n \times S_n \leq S_n \wr (\Omega_1, \Omega_2) S_2$.

Proof. First suppose $n \neq k$. Without loss of generality, $n > k$. By Lemma 3, it suffices to show that for any $\alpha, \beta \in \Omega$ and any $\varphi \in \text{Sym}(\Omega) \setminus (S_n \times S_k)$, we have $(\alpha, \beta) \in \langle S_n \times S_k, \varphi \rangle$. If $\alpha$ and $\beta$ lie in the same orbit $\Omega_1$, then $(\alpha, \beta) \in S_n \times S_k < \langle S_n \times S_k, \varphi \rangle$, and so now suppose we have $\alpha \in \Omega_1, \beta \in \Omega_2$. Since $\varphi \notin S_n \times S_k$, the permutation $\varphi$ must map some element of $\Omega_1$ to $\Omega_2$, and since $n > k$, not every element of $\Omega_1$ can be mapped to $\Omega_2$ in this way. Therefore there exist $\gamma, \delta \in \Omega_1$ such that $\varphi(\gamma) \in \Omega_2$ and $\varphi(\delta) \in \Omega_2$. Since we have $(\varphi(\gamma), \alpha), (\varphi(\delta), \beta) \in S_n \times S_k$, we may conjugate by these elements in order to assume without loss of generality that $\alpha = \varphi(\gamma)$ and $\beta = \varphi(\delta)$. Then, we have

$$(\alpha, \beta) = \varphi \circ (\gamma, \delta) \circ \varphi^{-1} \in \langle S_n \times S_k, \varphi \rangle,$$

as desired. Hence $S_n \times S_k$ is a maximal subgroup of $\text{Sym}(\Omega)$.

Now suppose that $n = k$. Then there is a permutation $\sigma \in \text{Sym}(\Omega) \setminus (S_n \times S_k)$ with $\sigma(\Omega_1) = \Omega_2$ and $\sigma(\Omega_2) = \Omega_1$. Then $\langle \Omega_1, \Omega_2 \rangle$ is a system of blocks for $(S_n \times S_k, \sigma)$, and so we have

$$S_n \times S_k < \langle S_n \times S_k, \sigma \rangle \leq S_n \wr (\Omega_1, \Omega_2) S_2.$$ 

This establishes the desired claim. Additionally, it is not hard to see the latter containment is in fact an equality, $\langle S_n \times S_k, \sigma \rangle = S_n \wr (\Omega_1, \Omega_2) S_2$. \hfill $\Box$

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3 The Affine Case

For $p$ prime, $\mathbb{Z}/p\mathbb{Z}$ is a field, and hence $(\mathbb{Z}/p\mathbb{Z})^k$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space for $k \geq 1$. Therefore we make the following definition.

**Definition 5.** For $p$ prime and $k \geq 1$, we define $\text{GL}_k(p) := \text{Aut}_{\mathbb{Z}/p\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^k)$, the group of vector space automorphisms of $(\mathbb{Z}/p\mathbb{Z})^k$.

Indeed, vector space automorphisms and (abelian) group automorphisms of $(\mathbb{Z}/p\mathbb{Z})^k$ coincide.

**Lemma 6.** For $p$ prime and $k \geq 1$, we have

$$\text{Aut}_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^k) = \text{GL}_k(p).$$

**Proof.** For $a \in \mathbb{Z}$, let $[a]_p$ denote the residue class of $a$ modulo $p$. Then for every $a \in \mathbb{Z}$ and $v \in (\mathbb{Z}/p\mathbb{Z})^k$, we have $av = [a]_p v$. Therefore, for any $\varphi \in \text{Aut}_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^k)$, we have

$$\varphi([a]_p v) = \varphi(av) = a\varphi(v) = [a]_p \varphi(v),$$

and so $\varphi \in \text{GL}_k(p)$, as desired. \qed

We define the **affine general linear group** to be the semidirect product of $(\mathbb{Z}/p\mathbb{Z})^k$ with $\text{GL}_k(p)$, where $\text{GL}_k(p)$ acts in the natural way on $(\mathbb{Z}/p\mathbb{Z})^k$.

**Definition 7.** For $p$ prime and $k \geq 1$, we define

$$\text{AGL}_k(p) := (\mathbb{Z}/p\mathbb{Z})^k \rtimes \text{GL}_k(p).$$

We must now make one more definition before we can state the affine case of the O'Nan-Scott Theorem.

**Definition 8.** If $G$ is a group, a subgroup $N \leq G$ is said to be a **minimal normal subgroup** of $G$ if $N$ is a normal subgroup of $G$ and does not properly contain any nontrivial normal subgroups of $G$.

**Theorem 9** (Affine Case). Let $\Omega$ be a set with $|\Omega| = N$, and let $G < \text{Sym}(\Omega)$ be such that $G$ acts primitively on $\Omega$. If $G$ has an abelian minimal normal subgroup, then $G$ is isomorphic to a subgroup of $\text{AGL}_k(p)$ for some prime $p$ and some $k \geq 1$.

In order to prove Theorem 9, we will require some lemmas and theorems. Where indicated, the proofs of these given here are slightly modified from the proofs given in [W09] and [DM96] in order to fit the presentation in this paper. In particular, we make explicit some steps that are elided in the source texts.

**Lemma 10** ([W09] 2.5). If a group $H$ acts primitively on $\Omega$, and $N$ is a nontrivial normal subgroup of $H$, then $N$ acts transitively on $\Omega$.

**Proof.** Suppose $N$ does not act transitively on $\Omega$, so that $\Omega$ is partitioned into orbits $N(\alpha_1), N(\alpha_2), \ldots, N(\alpha_k)$. Now consider $\alpha, \beta \in N(\alpha_i)$ for some $i$, so that $\alpha = n_1 \alpha_i$ and $\beta = n_2 \alpha_i$ for some $n_1, n_2 \in N$. Then by the normality of $N$, for any $h \in H$ there exist $n_1', n_2' \in N$ so that

$$gan_1 \alpha_i = n_1' g \alpha_i, \quad g \beta = gn_2 \alpha_i = n_2' g \alpha_i.$$  

Hence $g \alpha, g \beta \in N(g \alpha_i)$. It follows that $N(\alpha_1), N(\alpha_2), \ldots, N(\alpha_k)$ form a system of blocks for $H$, contradicting the assumption that $H$ acts primitively on $\Omega$. \qed

**Definition 11.** Given subgroups $K, H$ of a group $G$, we define the **commutator** $[K, H]$ of $K$ and $H$ to be the subgroup of $G$ generated by all elements of the form $h k h^{-1} k^{-1}$ for $k \in K, h \in H$.

**Lemma 12** ([W09] 2.6). If $N_1, N_2$ are distinct minimal normal subgroups of a group $H$, then $[N_1, N_2] = 1$.  

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Proof. The intersection $N_1 \cap N_2$ is trivial by the minimality of $N_1$, because it is a proper subgroup of $N_1$ that is normal in $H$. Since $[N_1, N_2] \leq N_1 \cap N_2$, we must have $[N_1, N_2] = 1$.

Definition 13. We say that a subgroup $H$ of a group $G$ is characteristic if $\varphi(H) = H$ for every automorphism $\varphi \in \text{Aut}(G)$. We say that a nontrivial group $G$ is characteristically simple if $G$ has no proper nontrivial characteristic subgroups.

Lemma 14 ([W09] 2.7). If $K$ is a characteristic subgroup of a group $N$ and $N$ is a normal subgroup of a group $H$, then $K$ is a normal subgroup of $H$.

Proof. For each $h \in H$, let $\varphi_h \in \text{Aut}(H)$ be given by $\varphi_h(g) = hgh^{-1}$. By the normality of $N$, we have $\varphi_h(N) = N$ for every $h \in H$, and hence $\varphi_h|_N \in \text{Aut}(N)$. Furthermore, since $K$ is characteristic in $N$, we have $\varphi_h(K) = \varphi_h|_N(K) = K$ for every $h \in H$, and hence $K$ is normal in $H$.

Lemma 15 ([W09] 2.8). If a group $G$ is characteristically simple, then $G$ is a direct product of isomorphic simple groups.

Proof. Suppose that $N$ is a minimal normal subgroup of $G$, and $\varphi \in \text{Aut}(G)$. Then for every $g \in G$,

$$g\varphi(N)g^{-1} = \varphi(\varphi^{-1}(g)N\varphi^{-1}(g^{-1})) = \varphi(N),$$

and so $\varphi(N)$ is normal in $G$. Furthermore, it is minimal normal, because if $K$ is subgroup of $\varphi(N)$ that is normal in $G$, then we may similarly see that $\varphi^{-1}(K)$ is a subgroup of $N$ that is normal in $G$. If $\varphi(N) \neq N$, then $\varphi(N) \cap N = 1$ by the minimality of $N$. By Lemma 12, it follows that $N\varphi(N) \cong N \times \varphi(N)$, because $[N, \varphi(N)] = 1$ implies that $\varphi(N)$ acts trivially on $N$ by conjugation. The groups $\varphi(N)$ for $\varphi \in \text{Aut}(G)$ generate $G$, because the subgroup $\langle \varphi(N) \mid \varphi \in \text{Aut}(G) \rangle$ generated by these $\varphi(N)$ is characteristic and $G$ is characteristically simple. Just as $N\varphi(N) \cong N \times \varphi(N)$, we likewise have

$$G = \langle \varphi(N) \mid \varphi \in \text{Aut}(G) \rangle \cong \prod_{\varphi \in \text{Aut}(G)} \varphi(N).$$

Since $G$ is a direct product of the $\varphi(N)$, any normal subgroup of any $\varphi(N)$ is also normal in $G$. Since each $\varphi(N)$ is minimal normal, it follows that each $\varphi(N)$ is simple. Indeed, these subgroups are all isomorphic to $N$, and so $G$ is a direct product of isomorphic simple groups, as desired.

Lemma 16 ([W09] 2.9). If $N$ is a minimal normal subgroup of a finite group $H$, then $N$ is direct product of isomorphic simple groups.

Proof. By Lemma 14, every characteristic subgroup of $N$ is normal in $H$. Since $N$ is minimal normal, it must therefore have no proper nontrivial characteristic subgroups. That is, $N$ is characteristically simple. By Lemma 15, we conclude that $N$ is a direct product of isomorphic simple groups.

Definition 17. We say that a group $N$ acts regularly on a set $\Omega$ if for every $\alpha, \beta \in \Omega$, there exists a unique $n \in N$ such that $n\alpha = \beta$.

Clearly regular group actions are transitive. One nice thing about regular group actions is the following lemma.

Lemma 18. Let $N$ be a group acting transitively on a set $\Omega$. The following are equivalent.

(i) $N$ acts regularly on $\Omega$.

(ii) Every $1 \neq n \in N$ has no fixed points.

(iii) $|N| = |\Omega|$.
Proof. (i) $\Rightarrow$ (ii): Suppose $N$ acts regularly on $\Omega$. Given $\alpha \in \Omega$, the element $1 \in N$ fixes $\alpha$, and hence by regularity is the unique element of $N$ that fixes $\alpha$. Hence every $1 \neq n \in N$ has no fixed points.

(ii) $\Rightarrow$ (iii): Suppose every $1 \neq n \in N$ has no fixed points, and fix some arbitrary $\alpha \in \Omega$. The map $f : N \rightarrow \Omega, n \mapsto n\alpha$ is surjective by the transitivity of $N$. Now suppose that $n\alpha = m\alpha$ for $n, m \in N$. Then $nm^{-1}$ has $\alpha$ as a fixed point, and hence $n = m$. Thus $f$ is injective, and hence bijective. Therefore $|N| = |\Omega|$.

(iii) $\Rightarrow$ (i): Suppose that $|N| = |\Omega|$, and let $\alpha \in \Omega$. Since $N$ acts transitively on $\Omega$, for each $\beta \in \Omega$ there is some $n_\beta \in N$ with $n_\beta \alpha = \beta$. Now,

$$|\{n_\beta \in N \mid \beta \in \Omega\}| = |\Omega| = |N|,$$

and so $N = \{n_\beta \in N \mid \beta \in \Omega\}$. Therefore, for each $\alpha, \beta \in \Omega$, there is at most one $n \in N$ such that $n\alpha = \beta$, and so $N$ acts regularly on $\Omega$.

Recall the following definition from group theory.

**Definition 19.** Given a subgroup $N$ of a group $H$, the centralizer of $N$ in $H$ is the subgroup

$$C_H(N) = \{h \in H \mid \forall n \in N, hnh^{-1} = n\}.$$

One nice thing about centralizers is that they satisfy the following property.

**Lemma 20.** If $N$ is a normal subgroup of a group $H$, then $C_H(N)$ is also normal in $H$.

**Proof.** Suppose $h \in H$ is such that $hnh^{-1} = n$ for every $n \in N$. Then for any $g \in H$, we have

$$ghg^{-1}ngh^{-1}g^{-1} = g^{-1}ngg^{-1} = n,$$

where the first equality holds because $g^{-1}ng \in N$ by the normality of $N$. Hence $ghg^{-1} \in C_H(N)$, and so $C_H(N)$ is normal in $H$.

**Lemma 21 (W09 2.10).** If a group $H$ acts primitively and faithfully on a set $\Omega$, and $N$ is a nontrivial normal subgroup of $H$, then either $C_H(N) = 1$ or $C_H(N)$ acts regularly on $\Omega$.

**Proof.** Suppose that $C_H(N) \neq 1$. Then by Lemma 10, $N$ and $C_H(N)$ act transitively on $\Omega$. If $\alpha \in \Omega$ is a fixed point of $1 \neq h \in C_H(N)$, then for any $n \in N$,

$$hn\alpha = nh\alpha = n\alpha,$$

and hence $N(\alpha)$ is contained in the set of fixed points of $h$. Since $N$ acts transitively, it follows that $h$ fixes every point of $\Omega$, and hence $h$ lies in the kernel of the action. Since $H$ acts faithfully, it follows that $h = 1$, contradicting our assumption.

**Lemma 22 (W09 2.11).** If a group $H$ acts primitively and faithfully on a set $\Omega$, and $N_1$ and $N_2$ are normal subgroups of $H$ with $[N_1, N_2] = 1$, then $N_2 = C_H(N_1)$ and $N_1 = C_H(N_2)$. In particular, $H$ has at most two minimal normal subgroups, and if it has an abelian normal subgroup, then it has only one minimal normal subgroup.

**Proof.** By Lemma 10, $N_1$ acts transitively on $\Omega$, and so $|N_1| \geq |\Omega|$, and by Lemma 21, $C_H(N_2)$ is regular, and hence $|C_H(N_2)| = |\Omega|$. Since $N_1 \leq C_H(N_2)$, we have $|N_1| \leq |C_H(N_2)| = |\Omega|$, and so $|N_1| = |C_H(N_2)|$. Since $N_1 \leq C_H(N_2)$ and $|N_1| = |C_H(N_2)|$, it follows that $N_1 = C_H(N_2)$. Similarly, one can see that $N_2 = C_H(N_1)$.

By Lemma 12, if $N_1$ and $N_2$ are any minimal normal subgroups of $H$, then $[N_1, N_2] = 1$, and hence every minimal normal subgroup other than $N_1$ is precisely equal to $C_H(N_1)$. Therefore $H$ has at most two minimal normal subgroups.

If $H$ has an abelian normal subgroup, then it has an abelian minimal normal subgroup $N$. Since $[N, N] = 1$, it follows that $N = C_H(N)$, and any minimal normal subgroup of $H$ is also precisely equal to $C_H(N) = N$. Hence $H$ has only one minimal normal subgroup.
Now let us suppose that $G < \text{Sym}(\Omega)$ is transitive, and let $H$ be the normalizer of $G$ in $\text{Sym}(\Omega)$. Let 

$$\Psi : H \to \text{Aut}(G)$$

be given by $\Psi(h) : g \mapsto h^{-1}gh$.

**Lemma 23 ([DM96] Theorem 4.2B).** With the above notation, let $\alpha \in \Omega$. If $\sigma \in \text{Aut}(G)$, then

$$\sigma \in \text{im}(\Psi) \iff \sigma(\text{Stab}_G(\alpha)) = \text{Stab}_G(\beta) \text{ for some } \beta \in \Omega.$$

**Proof.** $\Rightarrow$: Suppose that $\sigma \in \text{im}(\Psi)$, so that $\sigma = \Psi(h)$ for some $h \in H$. Then $\sigma(\text{Stab}_G(\alpha)) = h^{-1}\text{Stab}_G(\alpha)h = \text{Stab}_G(\alpha a)$. 

$\Leftarrow$: Suppose that $\sigma(\text{Stab}_G(\alpha)) = \text{Stab}_G(\beta)$ for some $\beta \in \Omega$. Let $t \in \text{Sym}(\Omega)$ be such that $\alpha = t\beta$. Then $\sigma(g) = t^{-1}gt$ for every $g \in G$. Since $\sigma \in \text{Aut}(G)$, it follows that $t \in H$. $\Box$

We only need to consider the case where $G$ is its own centralizer in $\text{Sym}(\Omega)$, and so we weaken [DM96] Corollary 4.2B with this additional assumption

**Lemma 24 ([DM96] Corollary 4.2B).** With the above notation, suppose that $G$ is regular and $G = C_{\text{Sym}(\Omega)}(G)$. Then $\text{im}(\Psi) = \text{Aut}(G)$, and $H \cong G \rtimes \text{Aut}(G)$.

**Proof.** Since $G$ is regular, $\text{Stab}_G(\alpha) = 1$, and so $\text{im}(\Psi) = \text{Aut}(G)$ by Lemma 23. Now, we have $G \cap \text{Stab}_H(\alpha) = 1$ since $G$ acts regularly. Since $G$ acts transitively, for any $h \in H$, we have some $g \in G$ with $gha = \alpha$, and so $h = g^{-1}(gh) \in G \text{Stab}_H(\alpha)$. Hence $H = G \text{Stab}_H(\alpha)$. Therefore we have $H = G \rtimes \text{Stab}_H(\alpha)$. Since $G = C_{\text{Sym}(\Omega)}(G) = \ker(\Psi)$, we have

$$\text{Aut}(G) = \text{im}(\Psi) \cong H/\ker(\Psi) = H/G \cong \text{Stab}_H(\alpha).$$

We conclude that $H \cong G \rtimes \text{Aut}(G)$. $\Box$

We remark that the choice of the letter $H$ to denote the normalizer of $G$ in $\text{Sym}(\Omega)$ is deliberate. The normalizer of a subgroup $G$ of a symmetric group is known as the holomorph of $G$. This terminology is explained by the (general case of) Lemma 24: The word “holomorph” contains the prefix “holo-,” meaning “whole,” along with “morph,” meaning “change.” By Lemma 24, $H$ contains $G$ along with the whole collection of ways of changing $G$, that is, $\text{Aut}(G)$.

We now prove Theorem 9.

**Proof of Theorem 9.** Suppose that $G < \text{Sym}(\Omega)$ such that $G$ acts primitively on $\Omega$, and suppose that $G$ has an abelian minimal normal subgroup $N$. By Lemma 16, $N$ is a direct product of isomorphic simple groups. The only finite abelian simple groups are of the form $\mathbb{Z}/p\mathbb{Z}$, and so there exists some prime $p$ and some $k \geq 1$ such that $N = (\mathbb{Z}/p\mathbb{Z})^k$. By Lemma 22, $N$ is the unique minimal normal subgroup of $G$ and satisfies $N = C_G(N)$, and by Lemma 21, $N$ acts regularly on $\Omega$. Let $H$ be the normalizer of $N$ in $\text{Sym}(\Omega)$; note that $G \leq H$ because $N$ is normal in $G$. By Lemma 24, we have that

$$H \cong N \rtimes \text{Aut}(N) = (\mathbb{Z}/p\mathbb{Z})^k \rtimes \text{GL}_k(p) = \text{AGL}_k(p).$$

Since $G < H$, we are done. $\Box$

This concludes the final case of the proof of Theorem 1. In order to classify all subgroups of finite symmetric groups, it now only remains to consider such groups that have nonabelian minimal normal subgroups. This will be done in the sequel to this paper.

**References**


The O’Nan-Scott Theorem II: The Full Theorem and its Applications
Bradley Zykoski

1 Introduction
1.1 Overview

Recall that the O’Nan-Scott Theorem gives us a classification of the subgroups of a finite symmetric group. We state the theorem here in full; any as yet undefined terminology will be defined in the course of this paper. The O’Nan-Scott Theorem admits several different formulations; we give the formulation as it appears in [W09].

**Theorem 1 (O’Nan-Scott).** If $H < S_n$ and $H \neq A_n$, then $H$ is a subgroup of one or more of the following:

(i) An intransitive group $S_k \times S_m$ with $n = k + m$,

(ii) An imprimitive group $S_k \wr S_m$ with $n = km$,

(iii) A primitive wreath product $S_k \wr S_m$ with $n = k^m$,

(iv) An affine group $AGL_d(p)$ with $n = p^d$,

(v) A group $T^m.(\text{Out}(T) \times S_m)$ of diagonal type with $n = |T|^{m-1}$,

(vi) An almost simple group.

In Part I [Z17] we proved cases (i), (ii), and (iv) of the above theorem, and gave the following definition.

**Definition 2.** For $p$ prime and $d \geq 1$, we define

$$AGL_d(p) := (\mathbb{Z}/p\mathbb{Z})^d \rtimes GL_d(p).$$

A group $H$ is affine if there exist $p, d$ such that $H \leq AGL_d(p)$.

Note that an affine group $H$ has a natural action on $(\mathbb{Z}/p\mathbb{Z})^d$, and that an affine subgroup of $S_n \cong \text{Sym}(\Omega)$ for $n = |\Omega| = |(\mathbb{Z}/p\mathbb{Z})^d| = p^d$ acts on $\Omega$ in a way that may be viewed as this natural action if we make an identification $\Omega \overset{\sim}{\rightarrow} (\mathbb{Z}/p\mathbb{Z})^d$ (This follows from the regularity of $N$ in Theorem 9 of Part I [Z17]). This justifies the claim that $n = p^d$ in (iv) of Theorem 1.

Let us introduce some terminology and notation that will be useful in the course of this paper.

**Definition 3.** Given a permutation group $H$ acting on a set $\Omega$, the **degree** of $H$ is $\deg H := |\Omega|$.

Hence the $n$ in Theorem 1 is the degree of the permutation group $H \leq \text{Sym} \{1, 2, \ldots, n\} = S_n$.

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1 This “viewing as” is made rigorous by the notion of permutation equivalence; see Definition 12.
Definition 4. If a group $G$ is an extension of groups $H$ and $K$, i.e. if there is some short exact sequence

$$1 \to H \to G \to K \to 1,$$

then we write $G = H.K$.

Therefore case (v) of Theorem indicates that there is a short exact sequence

$$1 \to T^m \to H \to \text{Out}(T) \times S_m \to 1.$$

1.2 Outline

In Section 2 we will give definitions for the undefined terminology involved in Theorem. In particular, we will define primitive wreath products, almost simple groups, and groups of diagonal type. In Section 3 we will make some remarks and observations concerning the proof of the O’Nan-Scott Theorem; we will not give the full proof. In Section 4 we will take a look at some of the applications of the O’Nan-Scott Theorem.

2 Further Types of Permutation Groups

2.1 Primitive Wreath Products

Given groups $H, K$ acting on finite sets $\Gamma, \Delta$, respectively, the wreath product $K \wr H$ has a natural imprimitive action on $\Gamma \times \Delta$. We will see here that, given mild hypotheses, $K \wr H$ also has a natural primitive action on $\Delta^\Gamma$. For the sake of brevity, let us denote group actions using exponential notation, so that e.g. $k \in K$ sends $\delta \in \Delta$ to the element denoted by $\delta^k$. Note that this is a right-action.

Definition 5. With the above notation, let $\Omega = \Delta^\Gamma$, $\varphi \in \Omega$, and $(f, h) \in K \wr H = K^\Gamma \rtimes H$. We define the product action of $K \wr H$ on $\Omega$ by

$$\varphi^{(f, h)}(\gamma) := \varphi^{(\gamma^{h^{-1}})}(f^{(\gamma^{h^{-1}})}).$$

In words, the $\gamma$-coordinate of $\varphi^{(f, h)}$ is given by the action of $f$ on the $\gamma^{h^{-1}}$-coordinate of $\varphi$.

The mild hypotheses for the primitivity of the product action are as follows.

Lemma 6 ([C99] Theorem 4.5, [DM96] Lemma 2.7A). With the above notation, the product action of $K \wr H$ on $\Omega$ is primitive if and only if $K$ acts primitively nonregularly on $\Delta$ and $H$ acts transitively on $\Gamma$.

To give a sense for how the product action works, we will give the proof of the necessity of some of these hypotheses for the primitivity of the product action.

Partial proof. Let $\varphi_\delta \in \Omega$ be the constant $\delta$-function $\varphi_\delta(\gamma) = \delta$ for every $\gamma \in \Gamma$. Let

$$L = \text{Stab}_{K \wr H}(\varphi_\delta).$$

If $H$ is not transitive and $\Sigma$ is an orbit of $H$ in $\Gamma$, then the subgroup

$$M = \{(f, 1) \in K \wr H \mid f(\gamma) \in \text{Stab}_K(\delta) \forall \gamma \in \Sigma\}$$

is normalized by $H$, and hence

$$L < MH < K \wr H,$$

2This is more or less verbatim the proof given in [DM96], although the proof of the necessity of the transitivity of $K$ has been simplified.
so that $L$ is non-maximal. Since a group is primitive if and only if its point-stabilizer is non-maximal, we conclude that $K \lhd H$ is not primitive.

If $K$ is not transitive, then neither is $K \lhd H$, because then the orbit $K \lhd H(\varphi_\delta)$ of $\varphi_\delta$ satisfies

$$K \lhd H(\varphi_\delta) \subseteq \{(f, h) \in K \lhd H \mid f(\gamma) \in K(\delta) \forall \gamma \in \Gamma\} \subseteq \Omega.$$  

Finally, if $K$ is transitive but not primitive, then there is some $R \leq K$ with $\text{Stab}_K(\delta) < R < K,$ and hence

$$L \leq \{(f, h) \in K \lhd H \mid f(\gamma) \in R \forall \gamma \in \Gamma\} < K \lhd H,$$

and so $K \lhd H$ is not primitive. 

It is worth noting that the product action is imprimitive when $\Gamma$ is infinite. The term primitive wreath product in case (iii) of Theorem 1 refers in particular to a wreath product acting primitively via the product action. Observe that the number $n = k^m$ is indeed the degree of $S_k \wr S_m$ as a primitive wreath product.

### 2.2 Almost Simple Groups

Let us first observe that a nonabelian simple group embeds into its automorphism group via the conjugation action.

**Lemma 7.** If $T$ is a nonabelian simple group, then $\varphi : T \to \text{Aut}(T), \varphi(t)u = tut^{-1}$ is injective.

**Proof.** Since $T$ is simple, either $\text{ker}(\varphi) = 1$ or $\text{ker}(\varphi) = T$. But $\text{ker}(\varphi) = Z(T)$ and $T$ is nonabelian, so $\text{ker}(\varphi) = Z(T) = 1$. 

Hence we may regard $T$ as a subgroup of $\text{Aut}(T)$. We may therefore make the following definition.

**Definition 8.** A group $G$ is called almost simple if there is a simple group $T$ such that $T \leq G \leq \text{Aut}(T)$.

An almost simple group $G$ with $T \leq G \leq \text{Aut}(T)$ has a natural action on the cosets of any maximal subgroup $M$ of $T$. Furthermore, we have the following proposition.

**Proposition 9.** The action of $G$ on the cosets of a maximal $M < T$ is primitive.

**Proof.** Clearly this action is transitive, and so it is primitive if and only if the stabilizer of the coset $eM$ is maximal. Certainly $M \leq \text{Stab}_G(eM)$, and if any $t \in T \setminus M$ stabilizes $eM$, then we also have $(M, t) \leq \text{Stab}_G(eM)$. Since $M$ maximal, it follows that $T = \text{Stab}_G(eM)$, and hence $M$ is normal in $T$. But then $M$ is a nontrivial normal subgroup of $T$, contradicting the simplicity of $T$. 

### 2.3 Groups of Diagonal Type

Let $T$ be a nonabelian simple group, $m \geq 1$. Then $T^m$ acts on the set $\Omega$ cosets of the diagonal subgroup

$$D = \{(t, t, \ldots, t) \mid t \in T\} \leq T^m$$

by left multiplication. Note that for any $(t_1, t_2, \ldots, t_m) \in T^m$, we have

$$(t_1, t_2, \ldots, t_m)D = (t_1t_m^{-1}, t_2t_m^{-1}, \ldots, 1)D.$$ 

Thus every coset of $D$ may be (uniquely) represented by an element of the form $(u_1, u_2, \ldots, u_{m-1}, 1)$. Therefore there is a natural bijection between $\Omega$ and the set $T^{m-1}$, and in particular $|\Omega| = |T|^{m-1}$.

Now, the action of $T^m$ on these coset representatives is given by

$$(t_1, \ldots, t_{m-1}, t_m)(u_1, \ldots, u_{m-1}, 1)D = (t_1u_1, \ldots, t_{m-1}u_{m-1}, t_m)D = (t_1t_m^{-1}, \ldots, t_{m-1}u_{m-1}t_m^{-1}, 1)D.$$
Observe that the copy of $D \leq T^m$ acts on $\Omega$ by inner automorphisms of $T$:

$$(t, \ldots, t)(u_1, \ldots, u_{m-1}, 1)D = (tu_1t^{-1}, \ldots, tu_{m-1}t^{-1}, 1)D.$$ 

Furthermore, there exists an action of $\text{Aut}(T)$ on $\Omega$ given by

$$\sigma(u_1, \ldots, u_{m-1}, 1)D = (\sigma(u_1), \ldots, \sigma(u_{m-1}), 1)D$$

for $\sigma \in \text{Aut}(T)$. There is also an action of $S_m$ on $\Omega$ given by

$$\eta(u_1, \ldots, u_{n-1}, u_n)D = (u_1^{\eta(1)}, \ldots, u_{n-1}^{\eta(n-1)}, u_n^{\eta(n)})D$$

for $\eta \in S_m$. Note that the actions by $\text{Aut}(T)$ and $S_m$ commute with each other. We say that a group $G$ is of diagonal type if it is an extension of the action of $T^m$ on $\Omega$ by some automorphisms of $T$ or symmetries in $S_m$. Since the action of $\text{Inn}(T)$ is already contained in the action of $T^m$ and since $\text{Aut}(T) = \text{Inn}(T).\text{Out}(T)$ (recall Definition [4]), any new actions on $\Omega$ by automorphisms of $T$ may be obtained by extending by outer automorphisms $\sigma \in \text{Out}(T)$. To be more precise, we have the following definition.

**Definition 10.** A group $G$ is of diagonal type if there is a nonabelian simple group $T$ and a positive integer $m$ such that

$$G = T^m.(\text{Out}(T) \times S_m).$$

The fact that the actions by $\text{Aut}(T)$ and $S_m$ commute with each other is captured in this definition by the fact that $G$ is an extension of $T^m$ by the direct product of $\text{Out}(T)$ and $S_m$. Observe that the number $n = |T|^m - 1$ in case (v) of Theorem [4] in is indeed the degree of $G$ as a group of diagonal type. We have now given definitions for all of the previously undefined terminology involved in Theorem [4].

### 3 The O’Nan-Scott Theorem

**3.1 Remarks on the Proof**

A major theme in the proof of the O’Nan-Scott Theorem is the analysis of minimal normal subgroups of the group $H$ in Theorem [4]. We have seen in Theorem 9 of Part I [Z17] that case (iv) of Theorem [4] arises precisely when the (unique) minimal normal subgroup of $H$ is abelian. Cases (iii), (v), and (vi) are also obtained upon analysis of minimal normal subgroups.

It is worth noting that many authors take the very similar approach of analyzing the socle of $H$, which is the subgroup generated by all minimal normal subgroups of $H$. In every case in which $H$ is primitive, it can be shown that the socle of $H$ is isomorphic to some $T^m$, where $T$ is a simple group and $m$ is a positive integer. Case (iv) is the unique case in which the $T$ is abelian, for indeed the socle and the unique minimal normal subgroup of $H$ coincide in this case. Hence in cases (iii), (v), and (vi), $T$ is nonabelian. Among these, (vi) is the unique case in which $m = 1$, and the socle of $H$ is precisely the nonabelian simple group $T$ for which $T \leq H \leq \text{Aut}(T)$.

**3.2 Alternative Formulations**

As remarked in §1.1, the O’Nan-Scott Theorem admits several different formulations. One common formulation (e.g. Theorem 4.8 of [C99]) which is obviously equivalent to that of Theorem [4] is as follows.

**Theorem 11 (O’Nan-Scott).** A non-alternating maximal subgroup of $S_n$ is isomorphic to one of groups in cases (i)-(vi) of Theorem [4].

Note that Theorem [11] does not claim that every group of one of the forms in cases (i)-(vi) is necessarily maximal. Indeed, Theorem 4 of Part I [Z17] demonstrates an example in which an intransitive group of the
form \( S_k \times S_k \) is a proper subgroup of a maximal imprimitive group of the form \( S_k \wr S_2 \). The O’Nan-Scott Theorem may also be stated for maximal subgroups of \( A_n \); these are just the intersections of \( A_n \) with groups of one of the forms in cases (i)-(vi).

Another formulation of the O’Nan-Scott Theorem is as a classification of the various types of primitive permutation groups. First, consider the following definition.

**Definition 12.** Two permutation groups \( G \leq \text{Sym}(\Omega) \) and \( H \leq \text{Sym}(\Delta) \) are said to be permutation equivalent if there are an isomorphism \( \varphi : G \to H \) and a bijection \( f : \Omega \to \Delta \) such that
\[
\varphi(g).f(\omega) = f(g.\omega)
\]
for every \( g \in G, \omega \in \Omega \).

Since every primitive permutation is permutation equivalent subgroup of some \( S_n \), we see that every primitive permutation group must be permutation equivalent to a subgroup of one of the groups in cases (iii)-(vi). This is a somewhat course classification, and the O’Nan-Scott Theorem is often stated in terms of a finer classification with more cases than these four; in particular see the paper [LPS88] and the blogpost [G09]. This formulation of the O’Nan-Scott Theorem lends itself particularly well to applications concerning primitive groups.

4 Applications

While the classification of primitive permutation groups is of great interest in its own right, it also serves to de-mystify these groups enough to make possible a wide variety of applications. The O’Nan-Scott Theorem is often applied in conjunction with another major classification theorem: the Classification of Finite Simple Groups (CFSG). This theorem is a result of the collective effort of many mathematicians over several decades. We do not attempt here to introduce all of the notions involved in the statement of the CFSG. In brief, the CFSG states that all but finitely many finite simple groups belong to one of three infinite families.

4.1 Degrees of Proper Primitive Groups

A primitive group is called proper if it neither symmetric nor alternating. By Theorem 1, any proper primitive group \( H \) that isn’t almost simple (case (vi)) either has degree \( |T|^\ell \) where \( T \) is a finite simple group and \( \ell \) is a positive integer (cases (iv) and (v)), or has degree \( \deg K^\ell \) where \( K \) is a primitive group and \( \ell \) is a positive integer (case (iii)). Thus, there are relatively few degrees for proper primitive groups. In fact, Cameron, Neumann, and Teague gave the following asymptotic estimate (see e.g. [C99] Theorem 4.12 or [DM96] Theorem 4.8A).

**Theorem 13** (Cameron-Neumann-Teague (1982)). Let \( e(x) \) denote the number of degrees \( n \leq x \) such that there is proper primitive group of degree \( n \). Then
\[
e(x) = 2\pi(x) + (1 + \sqrt{2})\sqrt{x} + O\left(\frac{\sqrt{x}}{\log x}\right),
\]
where \( \pi(x) \) is the prime counting function
\[
\pi(x) = |\{ p \in \mathbb{N} \mid p \text{ is prime and } p < x\}|
\]
and \( O(g(x)) \) denotes some function \( f(x) \) satisfying \( |f(x)| < cg(x) \) for some \( c \in \mathbb{R} \).

Following the discussion in §4.9 of [C99], we give some idea as to how the O’Nan-Scott Theorem and the CFSG are used in developing the above asymptotic estimate. As remarked above, many degrees of proper
primitive groups are of the form $n = k^\ell$ for some integer $\ell \geq 2$. For any integer $x$, the number of integers $n$ of this form with $n \leq x$ is equal to

$$\sum_{\ell=2}^{\infty} \left\lfloor \sqrt[\ell]{x} \right\rfloor = \sqrt[\ell]{x} + O(\sqrt[\ell]{x} \log x).$$

It is fairly easy to understand where the left-hand side of this equation comes from; the right-hand side is due to the fact that the left-hand sum must have at most $\log_2 x$ terms.

We now account for the cases where $n$ is not of the form $n = k^\ell$ for any $\ell \geq 2$, i.e. where $n$ has no integral roots of any order. Affine groups (case (iv)) have degree equal to either a nontrivial power a prime (and hence already counted above), or equal to a prime. This accounts for a $\pi(x)$ term in the expression for $e(x)$. Groups of diagonal type (case (v)) have degree equal to either a nontrivial power of some number (and hence already counted above), or equal to the order of a nonabelian simple group. Almost simple groups (case (vi)) are more difficult to analyze. The CFSG can be employed to obtain an estimate for these remaining two cases.

### 4.2 The Sims Conjecture

Let $G$ be a primitive group action on a set $\Omega$. We have the following definition.

**Definition 14.** A subdegree of of a permutation group $G \leq \text{Sym}(\Omega)$ is the length of an orbit of $\text{Stab}_G(\alpha)$ for some $\alpha \in \Omega$.

Suppose we want to know about permutation groups $G$ that have some $d > 1$ as a subdegree. There isn’t any clear way to bound $|G|$. For example (and indeed this is the example given in §4.8C of [DM96]), for $d = 2$, there are arbitrarily large primitive dihedral groups, and the reflection has orbits of length 2. However, it is possible to bound the order of $\text{Stab}_G(\alpha)$ for an arbitrary $\alpha \in \Omega$ (see e.g. [C99] Theorem 4.16 or [DM96] Theorem 4.8B).

**Theorem 15** (Sims Conjecture). There is a function $f$ such that if $G$ is a finite primitive permutation group with $d$ as a subdegree, then

$$|\text{Stab}_G(\alpha)| \leq f(d).$$

As in §4.1, it turns out that the most difficult case of this theorem is the case where $G$ is an almost simple group. Here one uses results by Thompson and Wielandt to obtain a function $g$ such that $\text{Stab}_G(\alpha)$ has a normal $p$-subgroup of index at most $g(d)$. The proof then requires the investigation of maximal normalizers of $p$-subgroups of $G$. This theorem was proved using the CFSG. We see a theme beginning to emerge here: the O’Nan-Scott Theorem allows one break conjectures about primitive groups into cases, dispense with the easier cases, and then use the CFSG to analyze the more difficult cases.

### 4.3 Length of $S_n$

**Definition 16.** The length $\ell(G)$ of a finite group $G$ is the length $k$ of the longest chain

$$G > G_1 > \cdots > G_{k-1} > G_k = 1$$

of subgroups of $G$.

By analyzing case-by-case the subgroups given in the O’Nan-Scott Theorem, the following formula is verified.

**Theorem 17** (Theorem 1.14 of [C99]). For $n$ a positive integer,

$$\ell(S_n) = \left\lceil \frac{3n}{2} \right\rceil - b(n) - 1,$$

where $b(n)$ denotes the number of 1’s in the base 2 representation of the number $n$. 

In §1.15 of [C99], a proof is sketched that there exists a chain in $S_n$ whose length is given by the right-hand side of the above equation. The power of the O’Nan-Scott Theorem and the CFSG are applied when showing that no longer chain exists. Let us first consider the following lemma.

**Lemma 18.** Let 
$$1 \to K \to G \overset{\pi}{\to} H \to 1$$
be an extension of groups. Then
$$\ell(G) \leq \ell(K) + \ell(H).$$

**Proof.** Consider a chain of subgroups
$$G > G_1 > \cdots > G_n > 1$$
in $G$. Then we have a (not necessarily strict) chain
$$K = G \cap K \geq G_1 \cap K \geq \cdots \geq G_n \cap K \geq 1$$
in $K$ and a (not necessarily strict) chain
$$H = \pi(G) \geq \pi(G_1) \geq \cdots \geq \pi(G_n) \geq 1$$
in $H$. Note that for any $k$, if $G_k \cap K = G_{k+1} \cap K$ and $G_k > G_{k+1}$, then $\pi(G_k) > \pi(G_{k+1})$, and similarly, if $\pi(G_k) = \pi(G_{k+1})$ and $G_k > G_{k+1}$, then $G_k \cap K > G_{k+1} \cap K$. Furthermore, if we have both $\pi(G_k) = \pi(G_{k+1})$ and $G_k \cap K = G_{k+1} \cap K$, then it follows that $G_k = G_{k+1}$. Hence we must have $\ell(G) \leq \ell(K) + \ell(H)$. \hfill $\square$

We proceed as in §4.11 of [C99]. We will not give a proof of Theorem [17] but we will at least indicate how the O’Nan-Scott Theorem is used in conjunction with Lemma [18] to get closer to the desired inequality $\ell(S_n) = \left\lceil \frac{3n}{2} \right\rceil - b(n) - 1$. Consider the function
$$f(n) = \left\lceil \frac{3n}{2} \right\rceil - b(n) - 1,$$
and suppose for the sake of induction that $\ell(S_m) = f(m)$ for all $m < n$. Let
$$S_n > G > \cdots > 1$$
be a chain in $S_n$ of maximal length $N$. Since $G$ is a maximal subgroup of $S_n$, it is isomorphic to one of the groups in cases (i)-(vi) of Theorem [1]. Many of these are expressed as group extensions, and so we may apply Lemma [18].

(i) In this case, $G = S_k \times S_m$ with $n = k + m$, and so
$$N = \ell(G) + 1 \leq \ell(S_k) + \ell(S_m) + 1 = f(k) + f(m) + 1$$
by Lemma [18] and the induction hypothesis.

(ii) In this case, $G = S_k \wr S_m$ with $n = km$, and so $N \leq mf(k) + f(m) + 1$.

(iii) In this case, $G = S_k \wr S_m$ with $n = k^m$, and so $N \leq mf(k) + f(m) + 1$.

(v) In this case, $G = T^m.(\text{Out}(T) \times S_m$ with $n = |T|^m-1$, and so $N \leq m\ell(T) + \ell(\text{Out}(T)) + f(m)$.

Further analysis is required to show that these upper bounds are themselves bounded above by $f(n)$, and to analyze the cases in which $G$ is not presented as a group extension. Let us conclude by remarking that there are several more interesting applications of the O’Nan-Scott Theorem, and many of these are surveyed in §4 of [C99] and §4 of [DM96].
References


https://symomega.wordpress.com/2009/10/12/the-onan-scott-theorem/

