A Polytopal Decomposition of Strata of Translation Surfaces

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Outline

1. Translation surfaces and strata
2. L^∞-Delaunay triangulations and isodelaunay polytopes
3. The infinite-adjacency phenomenon
4. Classifying and deleting the infinite adjacencies

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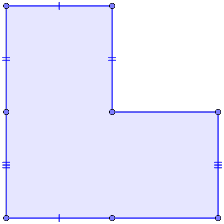

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Caveat: Throughout, we will talk about $\mathcal{H}(\kappa)$ when really we should be talking about an arbitrary finite orbifold cover of $\mathcal{H}(\kappa)$ that is a manifold.

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$\mathcal{H}(0)$, the stratum of flat tori with one marked point

The Stratum $\mathcal{H}(0)$ of Flat Tori

Each point $M \in \mathcal{H}(0)$ may be expressed as $M = \mathbb{R}^2/\Lambda$ for some lattice $\Lambda \subset \mathbb{R}^2$, where the marked point is the equivalence class of $0 \in \mathbb{R}^2$.

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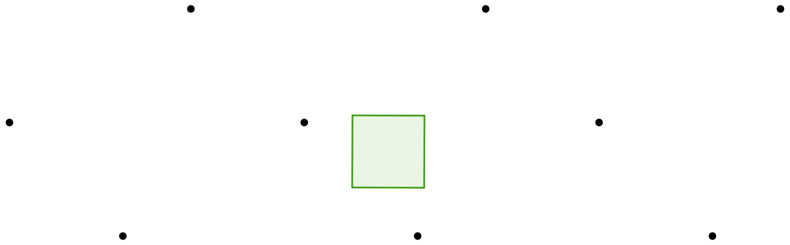

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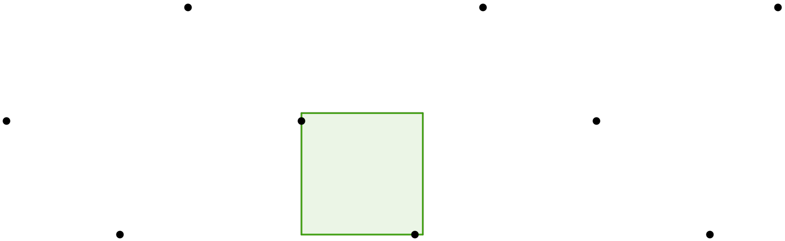

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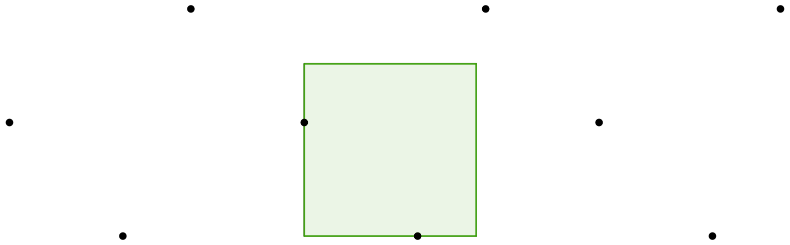

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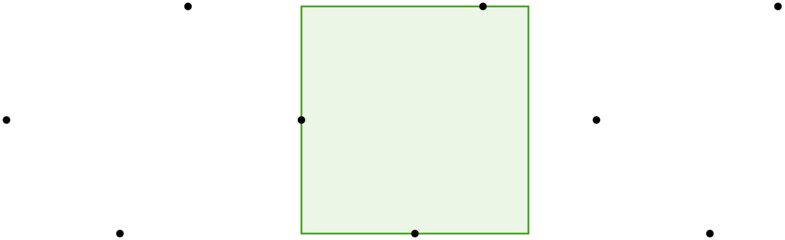

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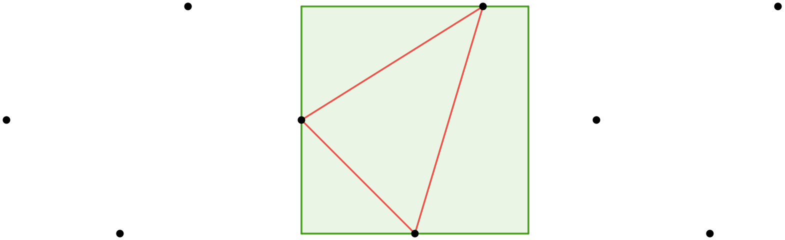

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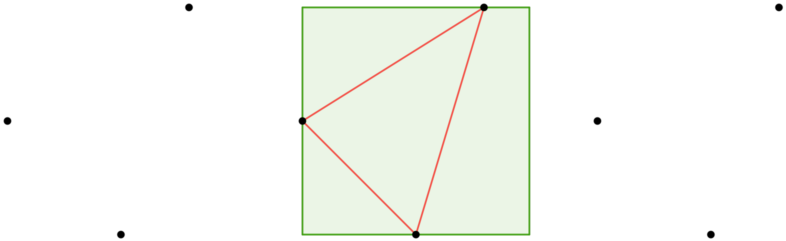

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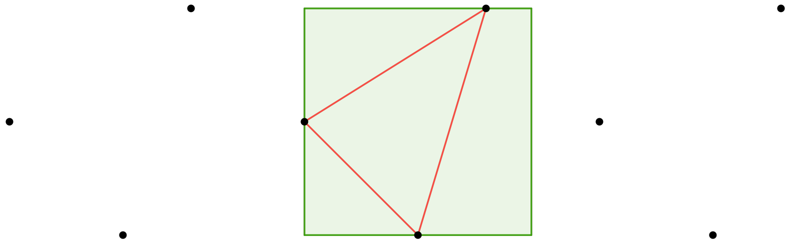

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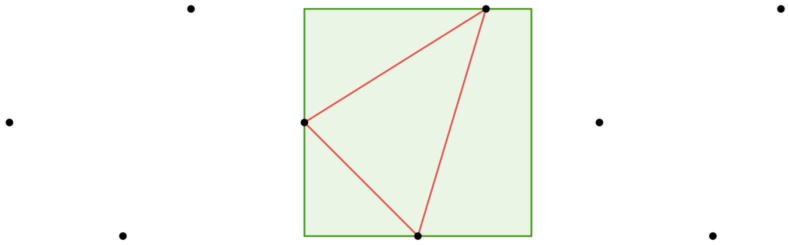

L^∞-Delaunay Triangulations

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If all the of the maximal singularity-free squares in a translation surface M have at most 3 singularities on their boundaries, then we may form line segments inside these squares connecting these singularities to form a triangulation of M.

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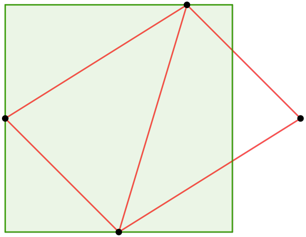

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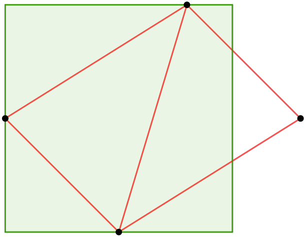
If all the of the maximal singularity-free squares in a translation surface M have at most 3 singularities on their boundaries, then we may form line segments inside these squares connecting these singularities to form a triangulation of M. We call this the L^∞-Delaunay triangulation of M if none of the edges are horizontal or vertical.



Degenerating Triangulations

If we begin with an L^∞-Delaunay triangulated surface M, we may obtain new translation surfaces by varying the edges of the triangulation. There are two ways in which the triangulation on the new surface may fail to be L^∞-Delaunay:

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- The lengths and widths of the $e \in \mathscr{B}$ form a system of local coordinates about $M \in \mathcal{H}(\kappa)$, called period coordinates.
- The condition that neither (i) nor (ii) above occurs is expressible as a system of linear inequalities in these coordinates (a polytope!).

Isodelaunay Polytopes

Let $\mathcal{D}(\kappa) \subset \mathcal{H}(\kappa)$ denote the set of translation surfaces that admit an L^∞-Delaunay triangulation. Then $\mathcal{D}(\kappa)$ is an open dense subset, and decomposes as a finite union of polytopes

$$\mathcal{D}(\kappa) = \coprod \mathcal{P},$$

where each $\mathcal{P}$ is a polytope given by the system of linear inequalities described previously, which guarantee [F18] that each translation surface $M \in \mathcal{P}$ has the "same" L^∞-Delaunay triangulation.

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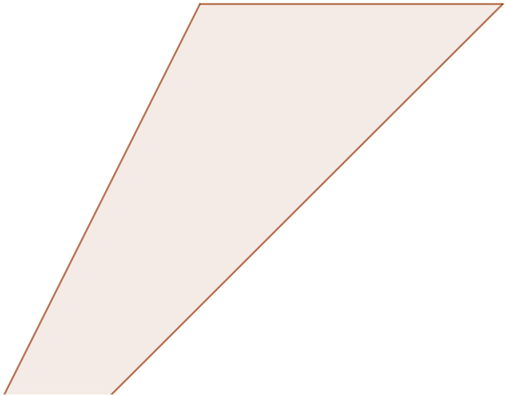
Definition (Isodelaunay polytopes)

We call each $\mathcal{P}$ an isodelaunay polytope in $\mathcal{H}(\kappa)$.

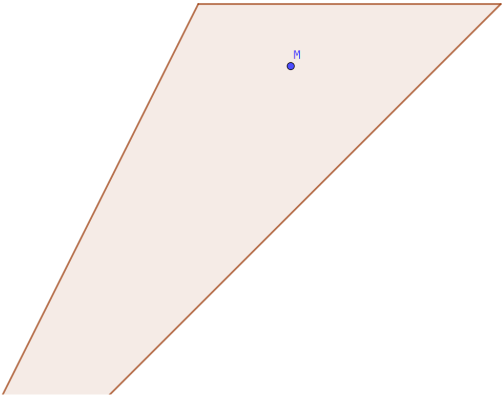
Adjacencies of Isodelaunay Polytopes

Pictured is one of the codimension 1 faces for an isodelaunay polytope $\mathcal{P}$ in $\mathcal{H}(0)$. What happens if we walk through it to the neighboring isodelaunay polytope $\mathcal{P}'$?



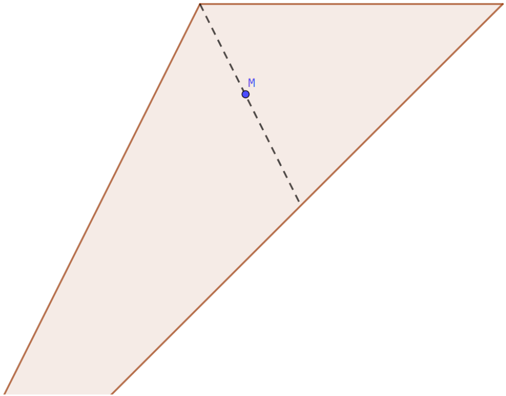
Adjacencies of Isodelaunay Polytopes

Usually we enter the neighboring polytope through a codimension 1 face of its own. That is to say, the surface $M \in \partial\mathcal{P} \cap \partial\mathcal{P}'$ fails precisely one of the inequalities that define $\mathcal{P}$, and also one of the inequalities that define $\mathcal{P}'$.



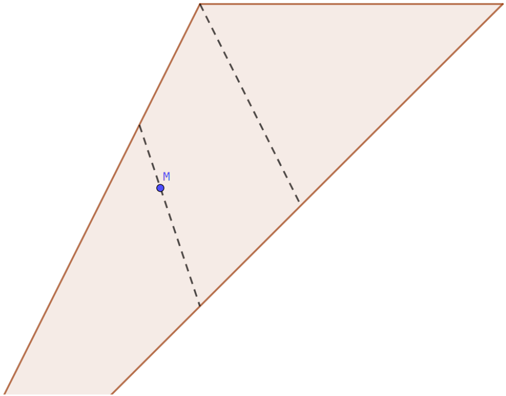
Adjacencies of Isodelaunay Polytopes

Sometimes, even though $M \in \partial\mathcal{P} \cap \partial\mathcal{P}'$ fails one of the inequalities that define $\mathcal{P}$, it actually happens to fail two of the inequalities that define $\mathcal{P}'$, and hence lies on a codimension 2 face of $\mathcal{P}'$.



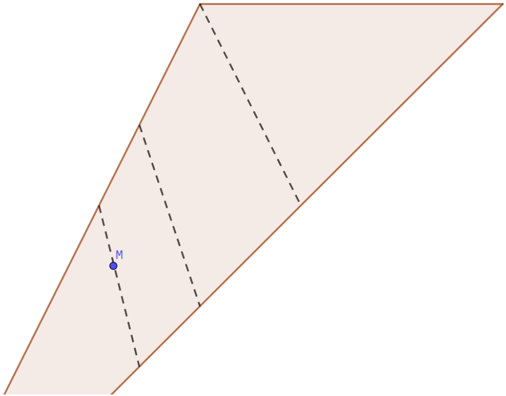
Adjacencies of Isodelaunay Polytopes

Traveling a bit further on, we find another segment where $M \in \partial\mathcal{P} \cap \partial\mathcal{P}'$ fails one of the $\mathcal{P}$-inequalities, but *two* of the $\mathcal{P}'$-inequalities. In fact, it is the same extra $\mathcal{P}'$-inequality that is failed both times.



Adjacencies of Isodelaunay Polytopes

We keep traveling, and we hit the same codimension 2 face of $\mathcal{P}'$ yet again! In fact, as we continue traveling down this unbounded wedge-shaped face of $\mathcal{P}$, we will hit that codimension 2 face infinitely many times!



The Infinite-Adjacency Phenomenon

Recall $\mathcal{D}(\kappa) = \coprod \mathcal{P}$ is open and dense in $\mathcal{H}(\kappa)$. It follows that

$$\mathcal{H}(\kappa) \setminus \mathcal{D}(\kappa) = \bigcup \partial \mathcal{P}$$

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$$\begin{aligned} \mathcal{H}(\kappa) \smallsetminus \mathcal{D}(\kappa) &= \bigcup \partial \mathcal{P} \\ &= \bigcup_{\mathcal{F} \text{ a face of some } \mathcal{P}} \mathcal{F} \end{aligned}$$

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The faces $\mathcal{F}$ subdivide each other along their adjacencies: The union $\bigcup \mathcal{F}$ may be refined to a disjoint union

$$\mathcal{H}(\kappa) \smallsetminus \mathcal{D}(\kappa) = \coprod \mathcal{F}',$$

where each polytope $\mathcal{F}'$ is is equal to a connected component of some maximal nonempty intersection of faces $\mathcal{F}$.

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The infinite-adjacency phenomenon

We have seen that, while there are finitely many $\mathcal{F}$, there may be infinitely many $\mathcal{F}'$. Frankel [F18] conjectured they can be classified in nice families.

Classifying the Infinite Adjacencies

$$\mathcal{H}(\kappa) \smallsetminus \mathcal{D}(\kappa) = \coprod \mathcal{F}'$$

Definition

A cylinder in a translation surface M is an isometric embedding $C \hookrightarrow M$, where

$$C = \mathbb{R}/w\mathbb{Z} \times (0, h). \qquad w, h > 0$$

The modulus of C is the quotient h/w.

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Theorem (Z, 2021)

For all but finitely many $\mathcal{F}'$, every $M \in \mathcal{F}'$ has a horizontal or vertical cylinder of modulus $\gg 1$.

Deleting the Infinite Adjacencies

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Let $\mathcal{C}(\kappa) \subset \mathcal{H}(\kappa)$ denote the union of faces $\mathcal{F}'$ that have a horizontal or vertical cylinder of modulus $\gg 1$. (A slight lie: we use a convenient combinatorial analogue instead of modulus)

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Idea of proof

Let's pick a number $\gg 1$, say 6, so that all $M \in \mathcal{C}(\kappa)$ have a horiz./vert. cylinder of modulus > 6. Let $\mathcal{H}_3(\kappa) \subset \mathcal{H}(\kappa)$ denote the set of translation surfaces where all cylinders have modulus ≤ 3. Then we have a deformation retraction of $\mathcal{H}(\kappa)$ onto $\mathcal{H}_3(\kappa)$ given by shortening cylinders of modulus > 3. We show that this retraction restricts to $\mathcal{H}(\kappa) \smallsetminus \mathcal{C}(\kappa)$, and hence this set also retracts onto $\mathcal{H}_3(\kappa)$.

A Polytopal Decomposition of Strata

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We have a homotopy equivalence $\mathcal{H}(\kappa) \simeq \mathcal{H}(\kappa) \smallsetminus \mathcal{C}(\kappa)$.

By these two theorems, we find an explicit model for the homotopy type of $\mathcal{H}(\kappa)$ as a finite union of polytopes

$$\mathcal{H}(\kappa) \smallsetminus \mathcal{C}(\kappa) = \coprod_{\substack{\text{no } M \in \mathcal{F}' \text{ has a horiz./vert.} \\ \text{cylinder of high modulus}}} \mathcal{F}',$$

which may be understood as a finite CW-complex minus some boundary subcomplex.

References

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F18 Frankel, Ian. “CAT(-1)-Type Properties for Teichmüller Space.” *arXiv preprint arXiv: 1808.10022* (2018).

Important references:

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- Guéritaud, François. “Veering Triangulations and Cannon-Thurston Maps.” *Journal of Topology* 9.3 (2016): 957-983.