Teichmüller Spaces are Complex Manifolds

Bradley Zykoski, Student Geometry/Topology Seminar, 3/26/19

Introduction: A topological surface S admits many non-biholomorphic complex structures (i.e. atlases of charts to $\mathbb{C}$ with holomorphic transition maps). How do I parametrize/topologize the space of all such structures? Today, we'll see a parameter space $\mathrm{Teich}(S)$ that's only slightly redundant and is a finite-dimensional complex manifold.

As is often the case in mathematics, we find it easier to analyze functions than to analyze objects, and so we rephrase the question "How do I measure the difference between two Riemann surfaces X, Y homeomorphic to S?" as "How do I measure how much a map $X \to Y$ fails to be a biholomorphism?"

Roadmap: (1) Measuring non-holomorphicity at a point

(2) Measuring non-holomorphicity globally

(3) Eliminating some redundancy: Passing to Teichmüller space

(4) Measuring non-Möbius-ness

(5) The Bers embedding gives complex coordinates on Teichmüller space

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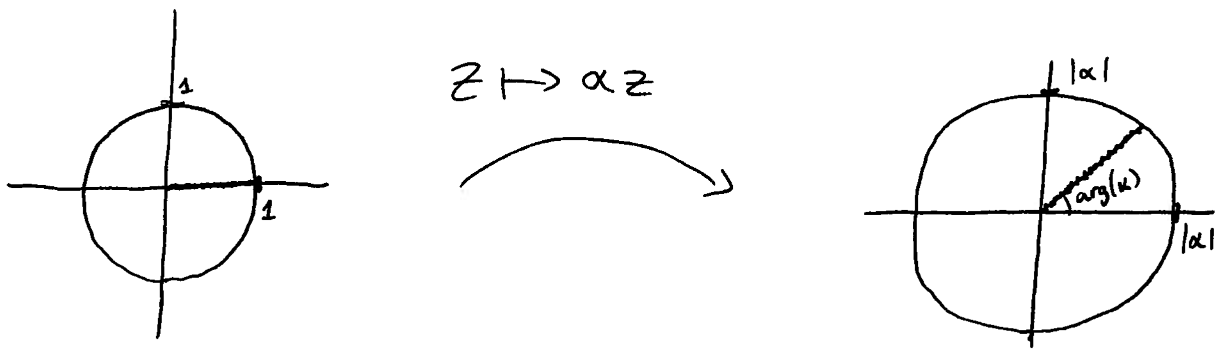
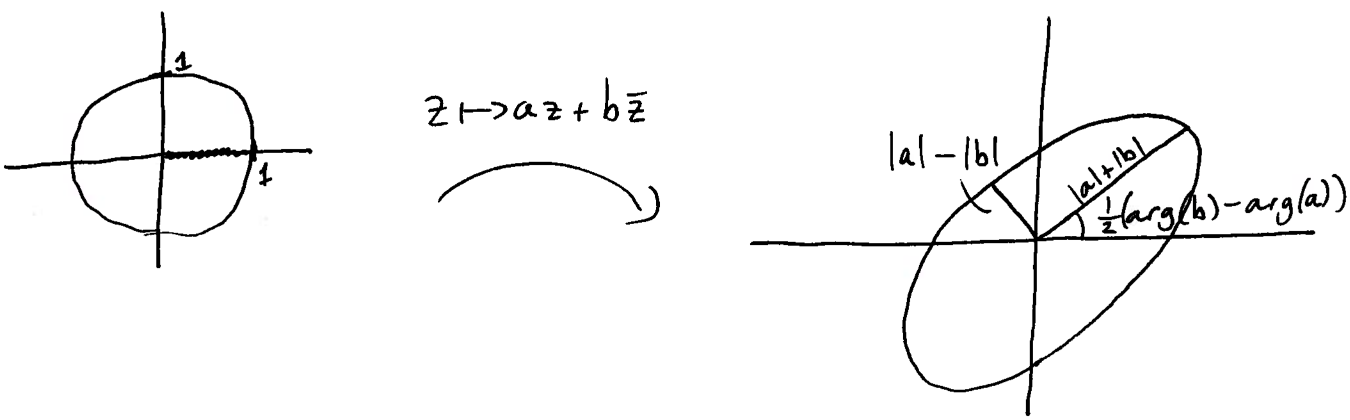
(1) Let $f: X \to Y$ be an orientation-preserving homeomorphism of Riemann surfaces. At a point $p \in X$, the question "How much does f fail to be a biholomorphism?" becomes "How much does $Df_p : T_p X \to T_{f(p)} Y$ fail to be $\mathbb{C}$-linear?"

$\mathbb{C}$-linear $\iff$ Maps circles to circles



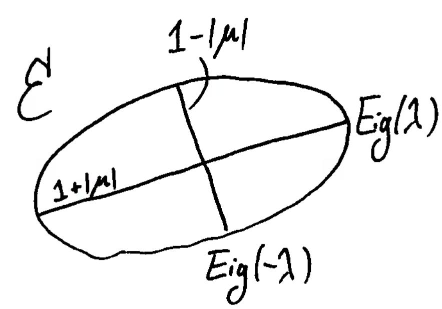
Failure to be $\mathbb{C}$-linear $\iff$ How much you stretch circles into ellipses



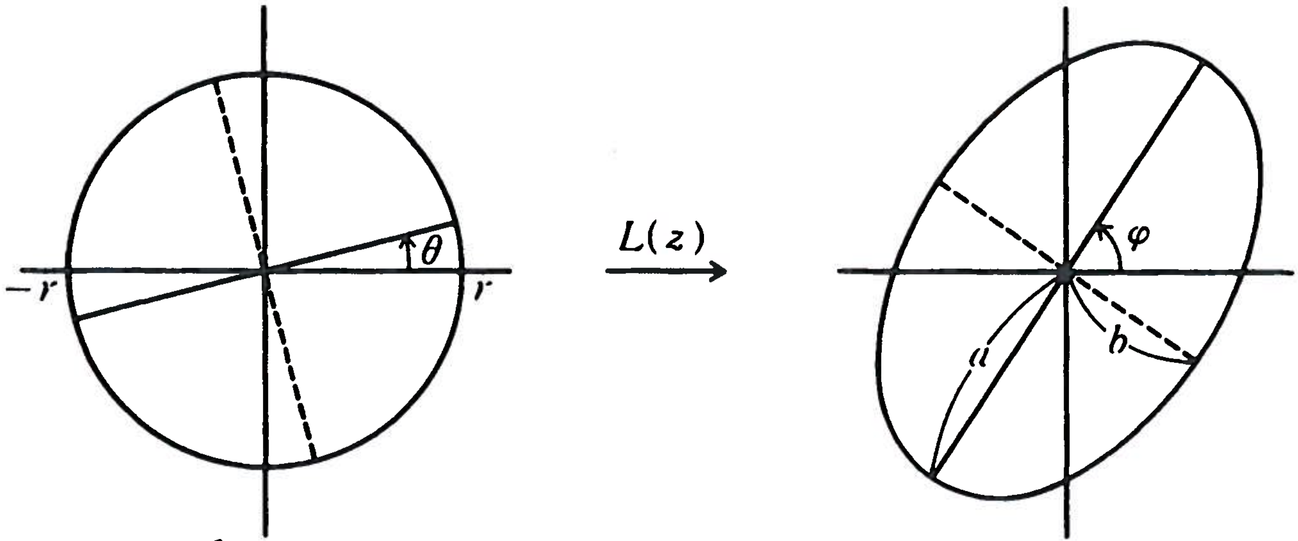
<u>Want</u>: A single object that captures all this stretching information

<u>Solution</u>: <u>Beltrami form</u>: $\mu(z \overset{A}{\mapsto} az+b\bar{z}) \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ is the map $z \mapsto \frac{b}{a}\bar{z}$.

<u>Properties</u>:
- $|\mu(A)| < 1$
- $\mu(A)$ is $\mathbb{C}$-antilinear $(\mu(A)(iz) = -i\mu(A)(z))$
- $\mu(A)$ has real and opposite eigenvalues $\pm\lambda$



- $\mu(B) = \mu(A)$ iff $B(\mathcal{E})$ is a circle.



$\theta = \frac{1}{2}\arg\mu(0)$

$\varphi = \theta + \arg f_z(0)$

$a = (1+|\mu(0)|)\,r\,|f_z(0)|$

$b = (1-|\mu(0)|)\,r\,|f_z(0)|$

(A) For $0 = p \in \mathbb{C}$, $L = Df_p = f_z(0)z + f_{\bar z}(0)\bar z$, L acts as pictured for $\mu(0) := \mu(L)$

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(2) As p ranges over all of X, we want to organize all of the Beltrami forms $\mu(Df_p)$ into a single object.

Definition: On a Riemann surface X, a <u>Beltrami differential</u> is a measurable section μ of the bundle $\mathrm{End}_{\mathbb{R}}(TX)$ that satisfies $\mu_p(iv) = -i\mu_p(v)$ $\forall v \in T_pX$, $\forall p \in X$ and $\operatorname{ess\,sup}_{p\in X} |\mu_p| < 1$. The space of Beltrami differentials on X is called $\mathrm{Bel}(X)$, and is an infinite-dimensional complex Banach manifold.

What sort of maps $f : U^{\subseteq \mathbb{C}} \to \mathbb{C}$ are "reasonably close to being biholomorphic onto their image?"

Definition: An orientation-preserving homeomorphism $f : U^{\subseteq\mathbb{C}} \to V^{\subseteq\mathbb{C}}$ is called <u>quasiconformal</u> if its distributional partial derivatives are locally L^2

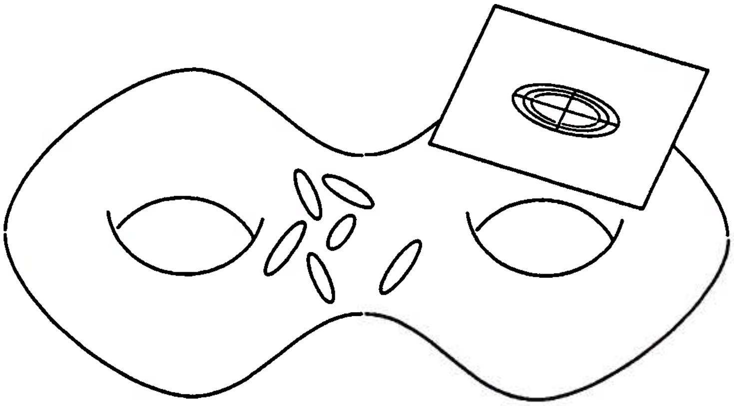


FIGURE 4.8.2 We think of a Beltrami ~~form~~ differential on a Riemann surface X as a measurable field of infinitesimal ellipses. The eccentricities of the ellipses are bounded, but the L^∞ nature of the field of ellipses means that they satisfy practically no correlation from point to point.

(B)

functions, and $\exists\, 0 \le k < 1$ such that

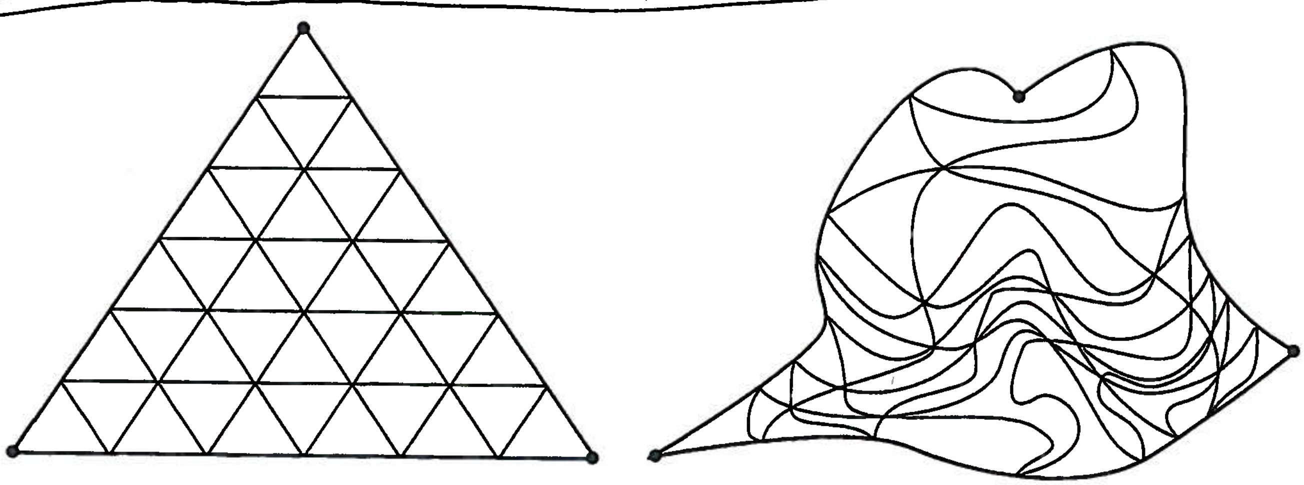
$$\left|\frac{\partial f}{\partial \bar z}\right| \le k \left|\frac{\partial f}{\partial z}\right| \quad \text{a.e.}$$

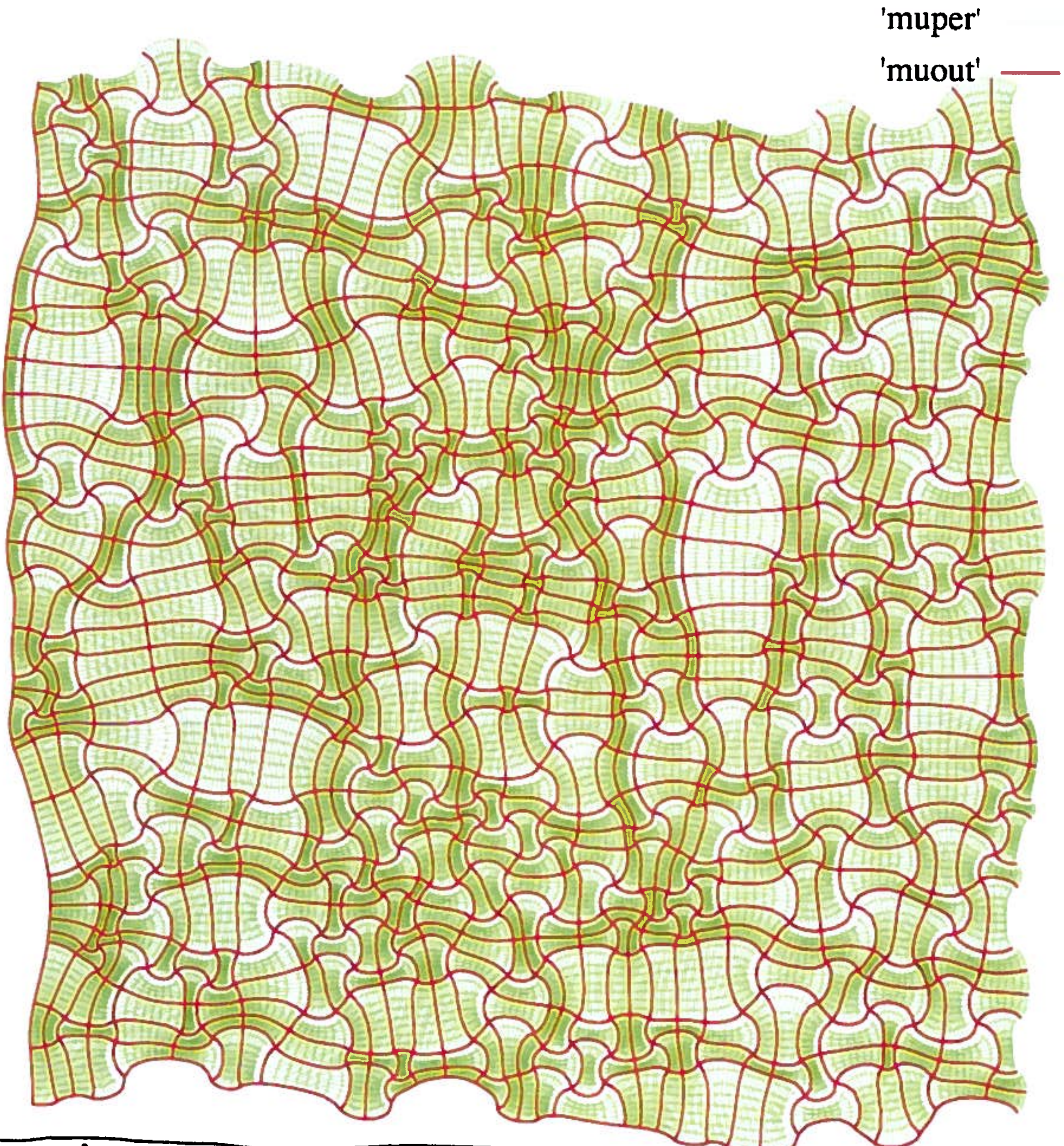
$$\Updownarrow$$

$$|\mu(Df_p)| \le k \quad \text{for a.e. } p \in U.$$

A map $f: X \to Y$ is called <u>quasiconformal</u> if it is quasiconformal when written in local coordinates.

(1C) The action of a quasiconformal map on a triangle

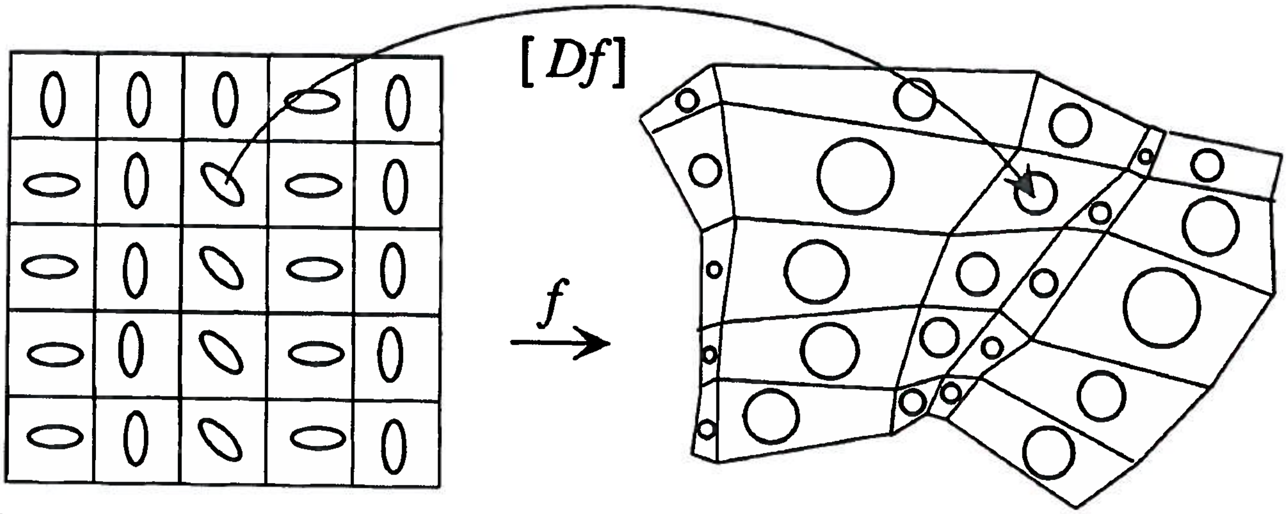




(D) The image of a random quasiconformal map whose domain is a square

For a quasiconformal map $f: X \to Y$ of Riemann surfaces, let $\mu(f) \in \mathrm{Bel}(X)$ be given by $\mu(f)_p := \mu(Df_p)$ at every $p \in X$.

We now are faced with an inverse problem: given some $\mu \in \mathrm{Bel}(X)$, does there exist a quasiconformal map $f: X \to Y$ for some Y with $\mu(f) = \mu$? Our definitions are precisely such that the answer is nice.



(E) A quasiconformal map with a prescribed Beltrami differential

<u>Measurable Riemann Mapping Theorem (MRM)</u>: Let X be a Riemann surface and let $\mu \in Bel(X)$. Then there exists a Riemann surface X_μ (unique up to biholomorphism) and there exists a quasiconformal map $f: X \to X_\mu$ such that $\mu(f) = \mu$. Every other quasiconformal $g: X \to X_\mu$ with $\mu(g) = \mu$ is of the form $g = \varphi \circ f$ for $\varphi \in Aut(X_\mu)$.

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(3) The MRM suggests that Beltrami ~~forms~~ differentials paramatrize the various complex structures on a fixed surface. Let us first define a space of Beltrami ~~forms~~ differentials that does not depend on a fixed complex structure on our closed topological surface S. This space is biholomorphic to $Bel(X)$ for any Riemann surface X homeomorphic to S.

<u>Definition</u>: Let S be a closed topological surface. We define

$$Bel(S) := \left\{ \left(S \xrightarrow[\text{homeo}]{h} \underset{\substack{\text{Riemann}\\ \text{surface}}}{X} \, , \, \mu \in Bel(X) \right) \right\} \Big/ \begin{pmatrix} (h,\mu) \sim (h',\mu') \\ \text{iff} \\ (h' \circ h^{-1})^* \mu' = \mu \end{pmatrix}$$

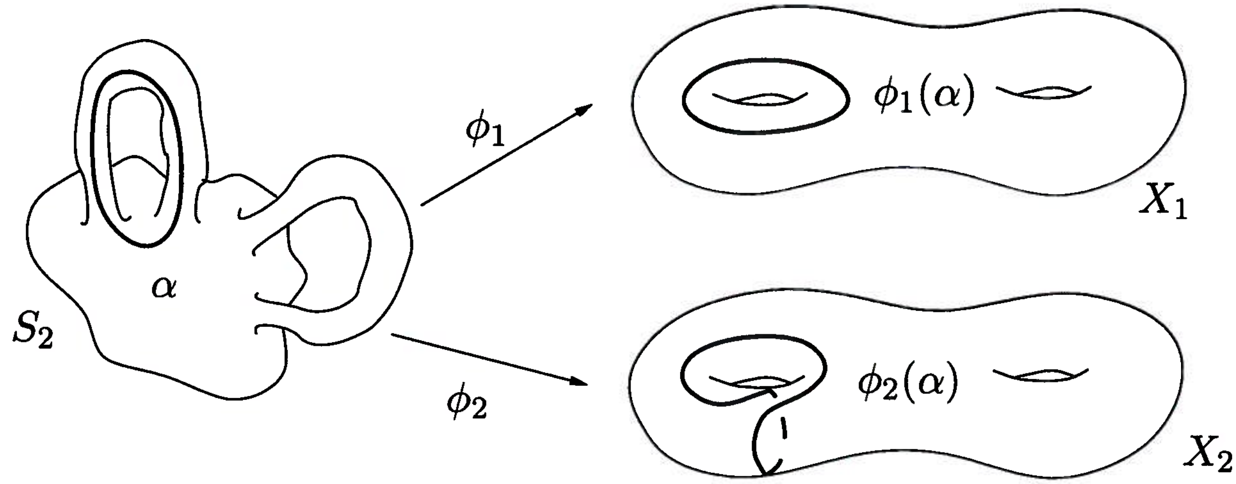

Figure 10.1 The hyperbolic surfaces X_1 and X_2 are isometric, but $\mathcal{X}_1 = [(X_1, \phi_1)]$ and $\mathcal{X}_2 = [(X_2, \phi_2)]$ are not the same point of $\mathrm{Teich}(S_2)$ since, for example, the way we have arranged things, $\ell_{\mathcal{X}_1}(\alpha)$ is not equal to $\ell_{\mathcal{X}_2}(\alpha)$.

$\boxed{(F)}$

Many of the points $(h, \mu) \in \mathrm{Bel}(S)$ induce the same complex structure X_μ on S, and $\mathrm{Bel}(S)$ is infinite-dimensional. We construct a quotient of $\mathrm{Bel}(S)$ that is much less redundant, and turns out to be a finite-dimensional complex manifold.

<u>Definition</u>: Let S be a closed topological surface. We define

$$\mathrm{Teich}(S) = \left\{ S \xrightarrow[\text{homeo}]{h} \underset{\substack{\text{Riemann} \\ \text{surface}}}{X} \right\} \Big/ \left(h \sim h' \text{ iff the diagram } \begin{tikzcd} & X \arrow[d, dashed, "\varphi \text{ biholomorphism}"] \\ S \arrow[ur, "h"] \arrow[r, "h'"'] & X' \end{tikzcd} \text{ exists and commutes up to homotopy} \right)$$

See (F) above

MRM gives us a surjection

$$\mathrm{Bel}(S) \twoheadrightarrow \mathrm{Teich}(S)$$

$$[S \xrightarrow{h} X, \mu \in \mathrm{Bel}(X)] \longmapsto [S \xrightarrow{h} X \xrightarrow{f} X_\mu] \quad \text{where } f \text{ is s.t. } \mu(f) = \mu.$$

(WEIRD!) Segue due to Bers: Lift $\mu \in Bel(X)$ along the universal covering $\begin{matrix} \mathbb{H}^2 \\ \downarrow \pi \\ X \end{matrix}$ to get $\pi^*\mu \in Bel(\mathbb{H}^2)$. Extend to $\hat{\mathbb{C}}$ via the formula

$$\hat{\mu} := \begin{cases} \pi^*\mu & \text{on } \mathbb{H}^2 \\ 0 & \text{elsewhere} \end{cases}$$

MRM gives $f^{\hat{\mu}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\mu(f^{\hat{\mu}}) = \hat{\mu}$. Normalize so that $f^{\hat{\mu}}$ fixes 0, 1, and ∞.

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(4) A linear-algebraic aside: If we fix bases, a linear map of vector spaces is equivalently a matrix of constants $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or a vector of linear homogeneous polynomials $\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$. So we define a cubic map of vector spaces to be a matrix of homogeneous quadratic polynomials or (equivalently) a vector of cubic homogeneous polynomials. In 1 dimension this equivalence just means $az^3 = (az^2)z$. The point is that if V is a 1-dimensional $\mathbb{C}$-vector space, a cubic map $V \to V$ is the same thing as a quadratic form $V \to \mathbb{C}$.

Definition: Let $f = \sum a_n z^n$ be a holomorphic map $U \overset{\subseteq \mathbb{C}}{\longrightarrow} \hat{\mathbb{C}}$. Let $M = \sum b_n z^n$ be the unique Möbius transformation agreeing with f up to degree 2. Then $D_p^3(f - M) := (a_3 - b_3) z^3$ is a cubic map $T_p U \to T_{f(p)} \hat{\mathbb{C}}$, and so $S_p(f) := 6\left((D_p f)^{-1} \circ D_p^3(f-M)\right) : T_p U \to T_p U$ is a cubic map. By the above discussion, we may equivalently view $S(f)$ as a (holomorphic) quadratic differential on $\mathcal{U}$.

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(5) By part (4), $S(f^{\hat{\mu}}|_{\overline{\mathbb{H}}^2})$ is a holomorphic quadratic differential on $\overline{\mathbb{H}}^2$, and it is not hard to see that it is invariant under the group Γ of deck transformations of the covering $\begin{matrix}\mathbb{H}^2\\ \downarrow\pi\\ X\end{matrix}$. (Note that these deck transformations are Möbius transformations, and hence extend to all of $\hat{\mathbb{C}}$.) Let $Q^\Gamma_\infty(\overline{\mathbb{H}}^2)$ denote the space of all such quadratic differentials, equipped with the L^∞-norm. By Riemann-Roch, $\dim_{\mathbb{C}} Q^\Gamma_\infty(\overline{\mathbb{H}}^2) = 3g-3$, where g is the genus of X.

Bers: Let $[S \xrightarrow{h} X] \in \mathrm{Teich}(S)$. Then

$$\Psi_h : \mathrm{Bel}(S) \to Q^\Gamma_\infty(\overline{\mathbb{H}}^2)$$
$$[S \xrightarrow{g} Y, \mu \in \mathrm{Bel}(Y)] \mapsto S\left(f^{\widehat{(g\circ h^{-1})^*\mu}}\Big|_{\overline{\mathbb{H}}^2}\right)$$

is holomorphic, and descends to an embedding $\Phi_h : \mathrm{Teich}(S) \to Q^\Gamma_\infty(\overline{\mathbb{H}}^2)$ (the Bers embedding).

$$\begin{tikzcd}
 & & B(0)_{\frac12} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \arrow[dll, bend right, dashed, "\sigma_h"'] \\
\mathrm{Bel}(S) \arrow[rr, "\Psi_h"] \arrow[dr] & & Q^\Gamma_\infty(\overline{\mathbb{H}}^2) \\
 & \mathrm{Teich}(S) \arrow[ur, "\Phi_h"'] &
\end{tikzcd}$$

<u>Ahlfors–Weill</u>: There exists a holomorphic section $\underset{\subseteq Q^{\Gamma}_{\infty}(\mathbb{H}^2)}{B_{\frac{1}{2}}(0)} \xrightarrow[\sigma_h]{} Bel(S)$ of Ψ_h on the ball $B_{\frac{1}{2}}(0)$ of radius $\frac{1}{2}$ about 0. (i.e. $\Psi_h \circ \sigma_h = Id_{B_{\frac{1}{2}}(0)}$).

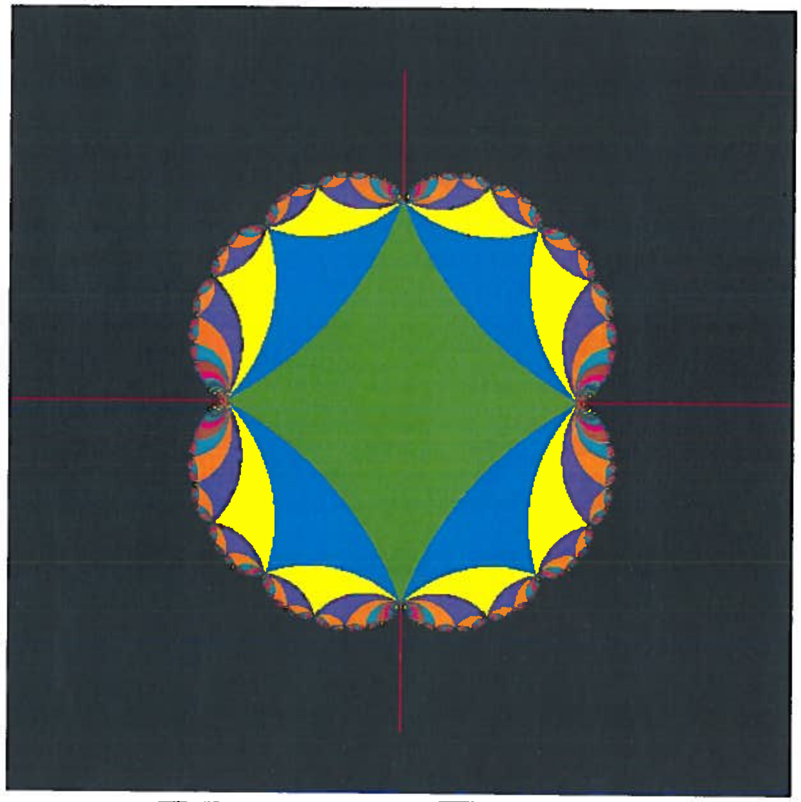
<u>Conclusion</u>: For $V_h := \Phi_h^{-1}(B_{\frac{1}{2}}(0))$, the <u>transition maps</u> on $V_h \cap V_{h'}$ are holomorphic.

Hence they define a complex manifold structure of dimension $3g-3$ on Teich(S).

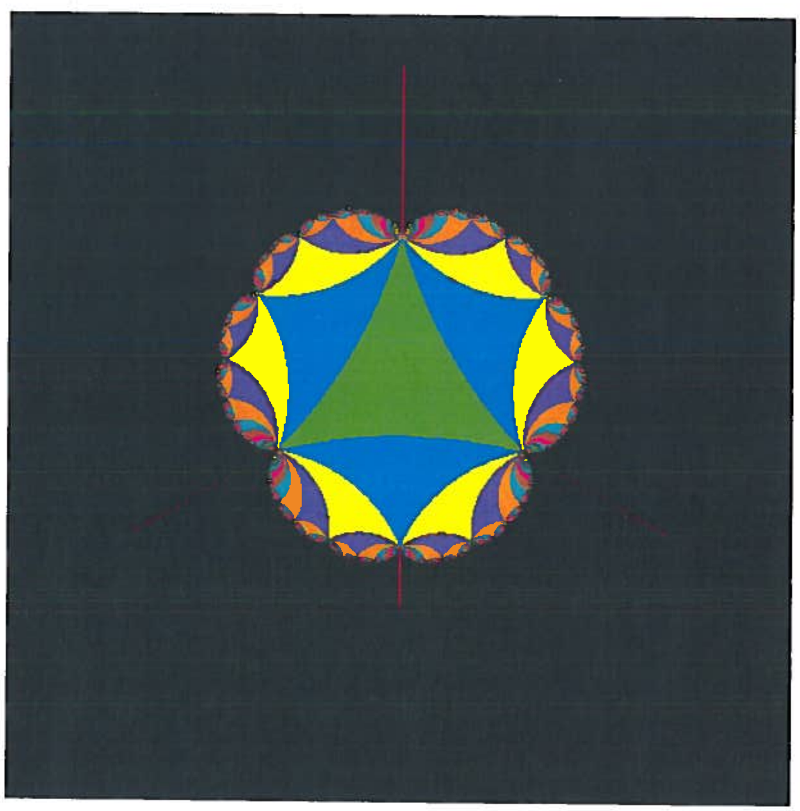
$$\hookrightarrow \Phi_{h'} \circ \Phi_h^{-1} = \Phi_{h'} \circ P \circ \sigma_h = \underbrace{\Psi_{h'}}_{\substack{\text{holomorphic}\\ \text{by Bers}}} \circ \underbrace{\sigma_h}_{\substack{\text{holomorphic by}\\ \text{Ahlfors–Weill}}}$$

$$\begin{tikzcd}
 & Bel(S) \arrow[d, "P"] & \\
 & Teich(S) & \\
V_h \arrow[ur, phantom, "\subseteq"] \arrow[d, "\Phi_h"] & & V_{h'} \arrow[ul, phantom, "\supseteq"] \arrow[d, "\Phi_{h'}"] \\
B_{\frac{1}{2}}(0) \arrow[uuur, dashed, bend left, "\sigma_h"] & & B_{\frac{1}{2}}(0) \arrow[uuul, dashed, bend right, "\sigma_{h'}"'] \\
Q^{\Gamma}_{\infty}(\mathbb{H}^2) \arrow[u, phantom, "\cap"] & & Q^{\Gamma'}_{\infty}(\mathbb{H}^2) \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

$\square$



(G) Image of the Bers embedding Φ_h of $\mathrm{Teich}(T^2)$ for h realizing T^2 as the square torus (above) and for h realizing T^2 as the hexagonal torus (below).



References

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Image Credits

(A) Imayoshi-Taniguchi, Page 18, Figure 1.11.

(B) Hubbard, Page 163, Figure 4.8.2.

(C) Hubbard, Page 140, Figure 4.5.5.

(D) Steffen Rohde, "Random QC map," available at www.math.harvard.edu/~ctm/gallery

(E) Hubbard, Page 150, Figure 4.6.1.

(F) Farb-Margalit, Page 264, Figure 10.1.

(G) "Bers slice - square torus" and "Bers slice - hex torus," available at www.math.harvard.edu/~ctm/gallery