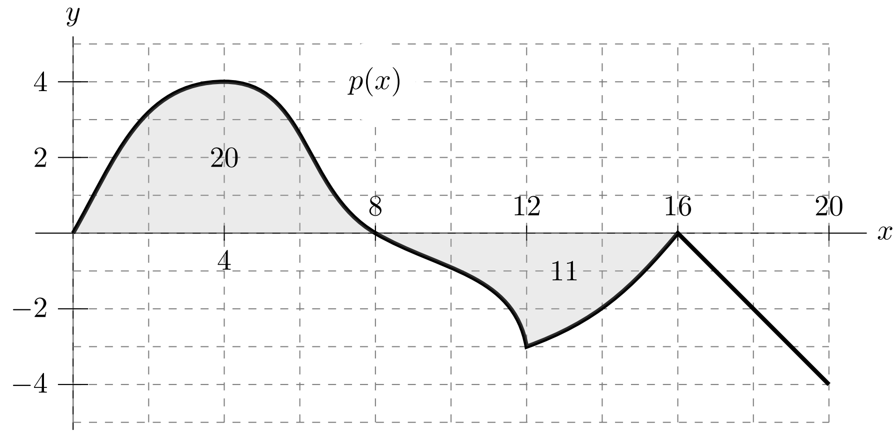
1. [12 points] Recall that a function h is odd if $h(-x) = -h(x)$ for all x. A portion of the graph of $p(x)$, an odd function, is shown below. Assume that the areas of the two shaded regions are 20 and 11, as indicated on the graph, and note that $p(x)$ is linear for $16 < x < 20$.



Remember to show your work throughout this problem.

a. [4 points] Compute the exact value of $\int_0^{20} (5 - 3p(x))\,dx$.

Solution: We have $\int_0^{20} (5 - 3p(x))\,dx = \int_0^{20} 5\,dx - 3\int_0^{20} p(x)\,dx$

$$= 100 - 3(20 - 11 - \frac{4 \cdot 4}{2}) = 97.$$

Answer: 97

b. [2 points] Compute the exact value of $\int_4^8 p'(x)\,dx$.

Solution: By the Fundamental Theorem, we have $\int_4^8 p'(x)\,dx = p(8) - p(4) = 0 - 4 = -4$.

Answer: -4

c. [3 points] Find the average value of $p(x)$ on the interval $-16 \le x \le 8$.

Solution: The average value is given by $\frac{1}{8-(-16)} \int_{-16}^{8} p(x)\,dx$. Since p is odd, we have $\int_{-8}^{8} p(x)\,dx = 0$, and $\int_{-16}^{-8} p(x)\,dx = -\int_8^{16} p(x)\,dx$. Thus, the average value is

$$\frac{1}{24}\left(\int_{-16}^{8} p(x)\,dx\right) = \frac{1}{24}\left(\int_{-16}^{-8} p(x)\,dx + \int_{-8}^{8} p(x)\,dx\right) = \frac{1}{24}\left(-\int_8^{16} p(x)\,dx + 0\right) = \frac{11}{24}.$$

Answer: $\frac{11}{24}$

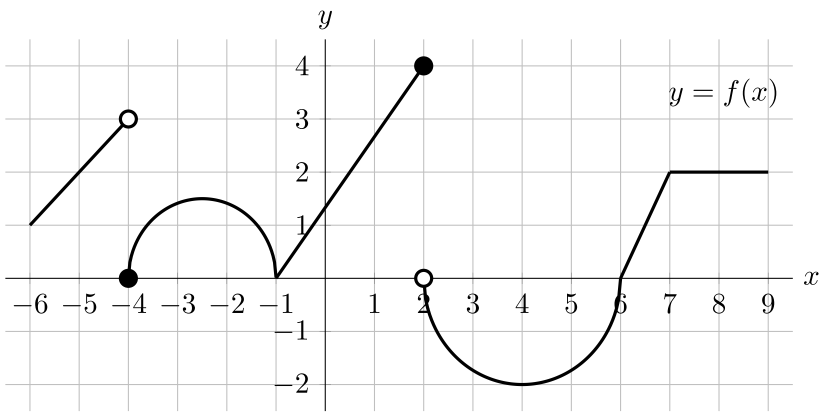
d. [3 points] Use a right Riemann sum with 3 equal subintervals to estimate $\int_{12}^{18} p(x)\,dx$. Write out all terms of the sum.

Solution:

$$2\,(p(14) + p(16) + p(18)) = 2(-2 + 0 + (-2)) = -8.$$

Answer: $\int_{12}^{18} p(x)\,dx \approx$ -8

2. [9 points] The graph of $f(x)$ shown below consists of lines and semicircles.



Use the graph above to calculate the answers to the following questions. Give your answers as exact values. You do not need to show work. If any of the answers can't be found with the information given, write "NEI".

a. [3 points] Find the average value of $f(x)$ on $[-4, 2]$.

Solution: $\frac{1}{6}\int_{-4}^{2} f(x)dx = \frac{1}{6}(\frac{1}{2}\pi(1.5)^2 + \frac{1}{2}(4)(3)) = \frac{1}{6}(\frac{9\pi}{8} + 6) = \frac{9\pi}{48} + 1$

b. [2 points] Find the value of $\int_4^9 |f(z)|dz$.

Solution: $\int_4^9 |f(z)|dz = -\int_4^6 f(z)dz + \int_6^9 f(z)dz = \frac{1}{4}\pi(2)^2 + \frac{1}{2}(3+2)(2) = 5 + \pi$

c. [2 points] Find the value of $4 < T \leq 9$ such that $\int_4^T f(x)dx = 0$.

Solution: We need to find a value of T for which

$$\int_4^T f(x)dx = \int_4^6 f(x)dx + \int_6^T f(x)dx = 0.$$

From the graph $\int_4^6 f(x)dx = -\pi$ and $\int_6^T f(x)dx = \frac{1}{2}((T-6)+(T-7))(2) = 2T - 13.$

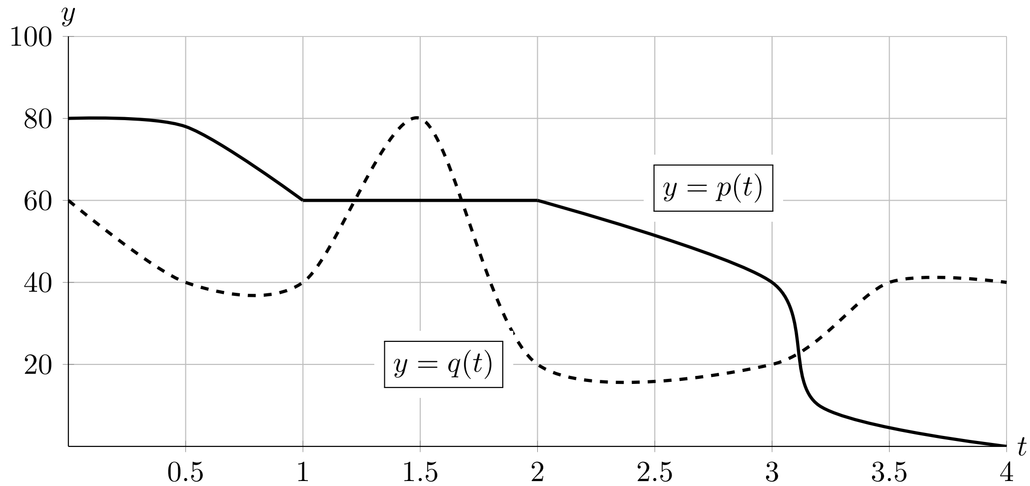
Solving for T on $2T - 13 = \pi$, we get $T = \frac{\pi + 13}{2}$.

d. [2 points] Find the value of $\int_{-8}^{-7} f(x+2) + 1\, dx$.

Solution:

$$\int_{-8}^{-7} f(x+2) + 1\, dx = \int_{-6}^{-5} f(x) + 1\, dx = \int_{-6}^{-5} f(x)dx + 1 = 1.5 + 1 = 2.5.$$

3. [7 points] At the cider mill, Xanthippe makes donuts fastest when she isn't distracted by customers. The rate, in donuts per hour, at which Xanthippe makes donuts t hours after 7 am is modeled by the function $p(t)$. Customers purchase donuts during their visit to the cider mill. The rate, in donuts per hour, at which customers purchase donuts t hours after 7 am is modeled by the function $q(t)$. The graphs of $y = p(t)$ (solid) and $y = q(t)$ (dashed) are shown below. Assume that at 7 am, Xanthippe begins with no donuts in stock.



a. [2 points] At what rate, in donuts per hour, is the number of donuts in stock (donuts produced but not yet sold) increasing/decreasing at 8:30 am? Be sure to circle one of INCREASING or DECREASING.

Solution: At $t = 1.5$, $p(t) - q(t) = -20$. The rate at which donuts are being sold exceeds the rate at which the donuts are being produced at a rate of 20 donuts/hr. Therefore, the number of donuts in stock is decreasing at a rate of 20 donuts/hr.

Answer: INCREASING $\boxed{\text{DECREASING}}$ at a rate of **20 donuts/hr**

b. [2 points] Write an expression involving p and q for the number of donuts in stock at 10 am. Your answer may involve definite integrals. Do not give approximations.

Solution: $p(t) - q(t)$ is the rate at which the number of donuts in stock is changing t hours after 7 am. By the fundamental theorem of calculus, $\int_0^3 p(t) - q(t)\, dt$ is the change in the number of donuts in stock between 7 am and 10 am. Since there were no donuts in stock at at 7 am, this is the number of donuts in stock at 10 am.

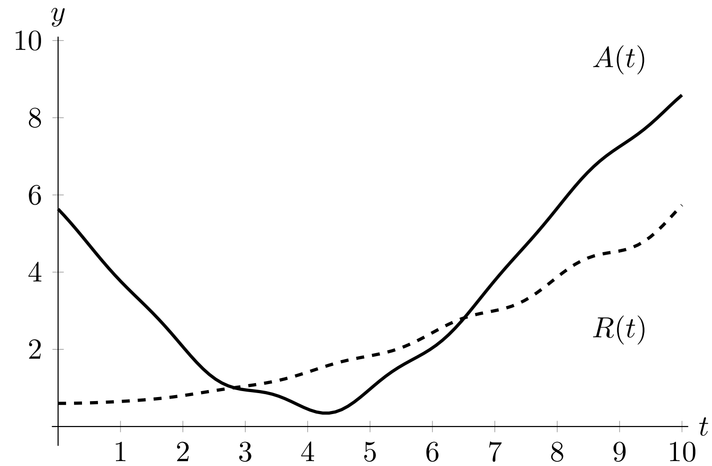
Answer: $\displaystyle\int_0^3 p(t) - q(t)\, dt$

c. [3 points] Xanthippe stops making donuts at 11 am. Assume that after 11 am, customers continue to purchase donuts at a constant rate of 40 donuts per hour until all of Xanthippe's donuts are sold out. Write an expression for the number of hours, starting at 11 am, that it takes for all her donuts to be sold out. Your answer may involve definite integrals. Do not give approximations.

Solution: The number of donuts in stock at 11 am is $\int_0^4 p(t) - q(t)\, dt$. When s hours have passed after 11 am, $40s$ donuts have been sold (assuming all donuts were not already sold), so we want to find s such that $40s = \int_0^4 p(t) - q(t)\, dt$.

Answer: $\displaystyle\frac{1}{40}\int_0^4 p(t) - q(t)\, dt$

8. [11 points] A tank contains 30 gallons of water. Beginning at 11 am, water is pumped in and out of the tank. Let $A(t)$ be the rate, in gallons per minute, at which the water is added into the tank t minutes after 11 am. Similarly, let $R(t)$ be the rate, in gallons per minute, at which the water is removed from the tank t minutes after 11 am. The graphs of the functions $A(t)$ (solid line) and $R(t)$ (dashed line) for $0 \le t \le 10$ are shown below.



a. [2 points] For which values of t is the total amount of water in the tank decreasing? Estimate your answer.

Solution: **Answer:** Approximately for $2.75 \le t \le 6.5$.

b. [1 point] At what time $0 \le t \le 10$ does the tank have the least amount of water?

Solution: **Answer:** $t = 0$.

In parts **c**. and **d**., give a mathematical expression that may involve $A(t)$, $R(t)$, their derivatives, and/or definite integrals.

c. [2 points] Find an expression for the total amount of water, in gallons, that was removed from the tank between 11:02 am and 11:05 am.

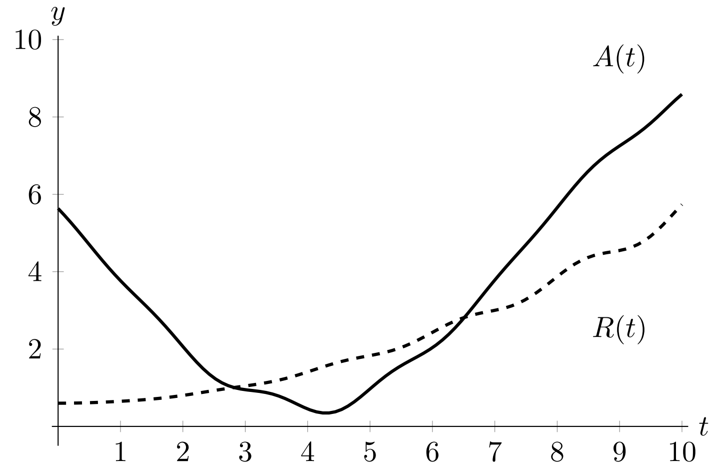
Solution: **Answer:** $\int_2^5 R(t)dt$.

d. [4 points] Find an expression for the amount of water, in gallons, in the tank at 11:10 am.

Solution: **Answer:** $30 + \int_0^{10} A(t) - R(t)dt$.

Problem continues on the next page

For your convenience, the graphs of $A(t)$ and $R(t)$ for $0 \le t \le 10$ are reprinted below.



e. [2 points] Suppose that there are 30 gallons of water in the tank at 11:20 am. Which of the following graphs could be the graph of $A(t)$ and $R(t)$ for $0 \le t \le 20$ in this case? Circle the *one* best answer.

Solution: