4. [13 points] One of the ways Captain Christina likes to relax in her retirement is to go for long walks around her neighborhood. She has noticed that early every Tuesday morning, a truck delivers butter to a local bakery famous for its cookie dough. Consider the following functions:

- Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of butter.
- Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter.
- Let $u(t)$ be the instantaneous rate, in pounds per hour, at which butter is being unloaded t hours after 4 am.

Assume that C, K, and u are invertible and differentiable.

a. [2 points] Interpret $K(C^{-1}(10)) = 20$ in the context of this problem.
Use a complete sentence and include units.

Solution: If the bakery spends \$10 on butter, then it can make 20 cups of cookie dough.

b. [3 points] Interpret $\int_5^{12} K'(b)\,db = 40$ in the context of this problem.
Use a complete sentence and include units.

Solution: 12 pounds of butter makes 40 cups more cookie dough than 5 pounds of butter does.

c. [2 points] Give a single mathematical equality involving the derivative of C which supports the following claim:
It costs the bakery approximately \$0.70 less to buy 14.8 pounds of butter than to buy 15 pounds of butter.

Answer: $C'(15) = 3.5$

d. [3 points] Give a single mathematical equality which expresses the following claim:
The number of pounds of butter unloaded between 5 and 8 am is twice as many as the bakery needs to make 5000 cups of cookie dough.

Answer: $\int_1^4 u(t)\,dt = 2K^{-1}(5000)$

e. [3 points] Assume that $u(t) > 0$ and $u'(t) < 0$ for $0 \le t \le 4$ and that $u(2) = 800$.
Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-IV.

I. 0 II. 800 III. $\int_1^2 u(t)\,dt$ IV. $\int_2^3 u(t)\,dt$

Solution: Since $u(t) > 0$, both integrals are greater than 0. Since $u'(t) < 0$, $u(t)$ is a decreasing function. Estimating $\int_1^2 u(t)\,dt$ with a right sum with one subdivision yields an underestimate of 800, and likewise, estimating $\int_2^3 u(t)\,dt$ with a left sum with one subdivision yields an overestimate of 800.

0 < $\int_2^3 u(t)\,dt$ < 800 < $\int_1^2 u(t)\,dt$

8. [11 points] The energy, in megajoules (MJ), produced by a wind turbine depends on the speed of the wind. In particular, suppose $P(s)$ is the power, in megajoules per hour (MJ/h), produced by the turbine when the speed of the wind is s kilometers per hour (km/h). Also suppose that $W(t)$ gives the wind speed, in km/h, at the turbine's location t hours after noon on a typical day.

Assume that $P(s)$ is invertible, and that both $P(s)$ and $W(t)$ are differentiable.

a. [2 points] Give a practical interpretation of the equation $P(W(0)) = 8$.

Solution: At noon on a typical day, the turbine produces 8 MJ/h of power.

b. [3 points] Give a practical interpretation of the equation $\int_0^5 P(W(t))\,dt = 46$.

Solution: From noon to 5 p.m. on a typical day, the turbine generates 46 MJ of energy.

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(4) = 21$$

From 4 pm to 4:10 pm, . . .

Solution: the wind speed at the turbine's location increases by approximately 3.5 km/h.

d. [3 points] Circle the one statement below that is best supported by the equation

$$(P^{-1})'(13) = 2.9.$$

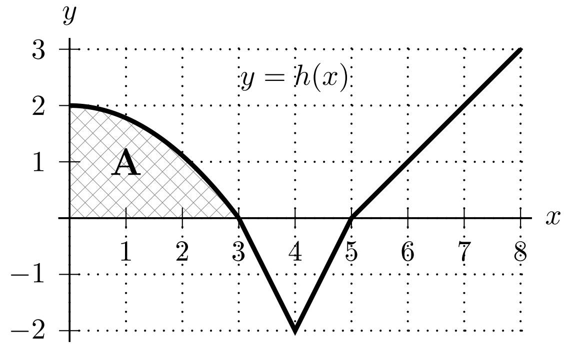
i. *If the turbine is producing 13 MJ/h of power, the wind speed must increase by approximately 2.9 km/h to produce an additional MJ/h of power.*

ii. If the wind is blowing at 13 km/h and increases to 14 km/h, the power produced by the turbine will increase by about 2.9 MJ/h.

iii. If the wind speed is 13 km/h, the power generation of the turbine will increase by one MJ/h if the wind speed increases to about 15.9 km/h.

iv. When the turbine is generating 13 MJ/h of power, an increase of one km/h in wind speed will produce approximately 2.9 MJ/h more power.

1. [13 points]



The graph of a function $h(x)$ is shown on the right. The area of the shaded region A is 4, and $h(x)$ is piecewise linear for $3 \le x \le 6$.

Compute each of the following. If there is not enough information to compute a value exactly, write NOT ENOUGH INFO.

a. [2 points] Find $\int_0^3 (h(x) + 2)\ dx$.

Solution:
$$\int_0^3 (h(x)+2)\ dx = \int_0^3 h(x)\,dx + \int_0^3 2\,dx = 4 + (3)(2) = 10\,.$$

Answer: $\int_0^3 (h(x)+2)\ dx =$ 10

b. [2 points] Find the average value of $h(x)$ on the interval $[0, 4]$.

Solution: The average value is given by $\dfrac{1}{4-0}\displaystyle\int_0^4 h(x)\,dx = \frac{1}{4}(4-1) = \frac{3}{4}\,.$

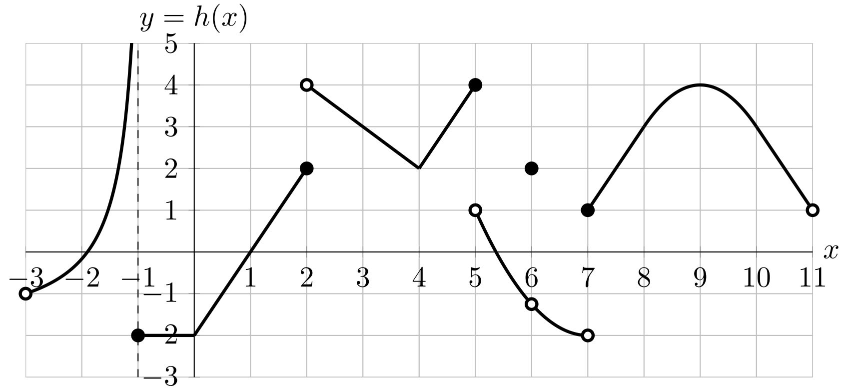
Answer: $\frac{3}{4}$

c. [3 points] Let $J(x) = \sin(\pi h(x))$. Find $J'(3.5)$.

Solution: Since $J'(x) = \cos(\pi h(x))\pi h'(x)$, we have that
$$J'(3.5) = \cos(\pi h(3.5))\pi h'(3.5) = \cos(-\pi)\pi(-2) = 2\pi\,.$$

Answer: $J'(3.5) =$ 2π

1. [13 points] The graph of a portion of a function $y = h(x)$ is shown below. Note that the graph is linear where it appears to be linear, including on the intervals $[7, 8]$ and $[10, 11)$.



a. [2 points] At which of the following points p is $h(x)$ *not* continuous at $x = p$? Circle *all* such values.

Solution: [p=-1] $p = 1$ [p=2] $p = 4$ [p=5] NONE OF THESE

b. [2 points] For which of the following values a is $\lim_{x \to a^+} h(x) = h(a)$? Circle *all* such values.

Solution: [a=-1] $a = 2$ [a=4] $a = 5$ $a = 6$ NONE OF THESE

For parts **c.**–**e.**, find the exact value of each of the expressions. If the value does not exist, write DNE. If there is not enough information, write NI.

c. [2 points] Calculate the average value of $h(x)$ on the interval $[-1, 1]$.

Solution:

$$\frac{1}{1-(-1)} \int_{-1}^{1} h(x)dx = \frac{1}{2} \int_{-1}^{1} h(x)dx = \frac{1}{2}(-3) = -1.5.$$

Answer= -1.5.

d. [4 points] Suppose $g(x) = h(3h(x))$. Calculate $g'(1.5)$. Show all your computations to receive full credit.

Solution:

$$g'(x) = h'(3h(x))(3h(x))' = 3h'(3h(x))h'(x).$$

Then $g'(1.5) = 3h'(3h(1.5))h'(1.5) = 3h'(3(1))(2) = 6h'(3) = 6(-1) = -6$

Answer= -6.

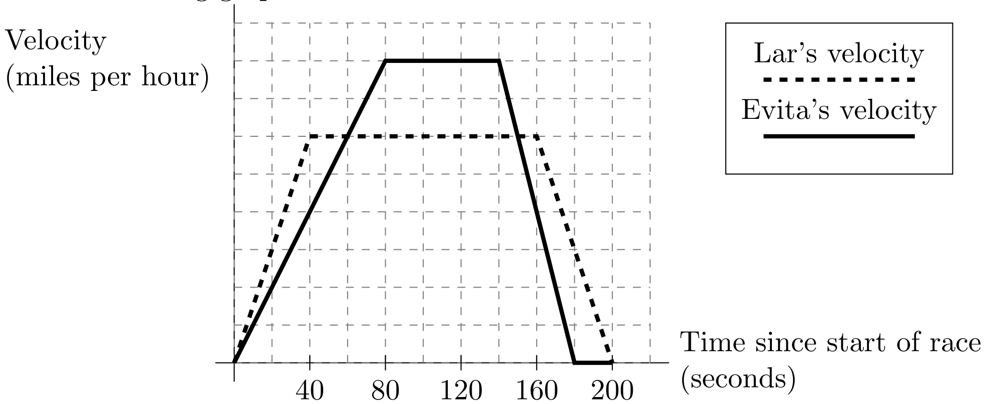
e. [3 points] Calculate $\int_{7.5}^{10.5} h''(x)\, dx$.

Solution: Using the Fundamental Theorem of Calculus we obtain

$$\int_{7.5}^{10.5} h''(x)\, dx = h'(10.5) - h'(7.5) = (-2) - (2) = -4.$$

Answer= -4.

3. [12 points] Lar Getni and Evita Vired run a half-mile race. After the race, C.T. Latnem Adnuf receives the following graph of the two runners' velocities over the course of the race.



Unfortunately, whoever made the graph forgot to label the scale of the vertical axis, and C.T. needs your help to answer the following questions. You may assume that the horizontal grid lines are evenly spaced, but do not assume that the scales of the two axes are the same. You may also assume that both runners completed the race and then stopped running.

a. [1 point] Who won the race?

Answer: **Evita**

b. [2 points] During what time interval(s) was Lar ahead of Evita?

Answer: $0 < t < 100$

c. [2 points] During what time interval(s) was Lar running faster than Evita?

Answer: $0 < t < 60$ **and** $150 < t < 200$

d. [4 points] What was the maximum speed (in miles per hour) attained by Lar? By Evita? *Remember to show your work.*

Solution: There are 48 boxes under the graph of Lar's (and also of Evita's) velocity. Let c denote the vertical dimension of a box, in miles per hour. The horizontal dimension of a box is 20 seconds, or $\frac{20}{3600}$ hours. Since the race is $\frac{1}{2}$ a mile long, and the area under the curve is equal to the distance traveled, we must have

$$48 \cdot c \cdot \frac{20}{3600} = \frac{1}{2}$$

so $c = 1.875$. Thus, the maximum speed attained by Lar is $6 \cdot 1.875 = 11.25$ mph, and the maximum speed attained by Evita is $8 \cdot 1.875 = 15$ mph.

Answer: Lar's max speed: **11.25 mph** and Evita's max speed: **15 mph**

e. [3 points] Let $v(t)$ (respectively, $w(t)$) be Evita's (respectively, Lar's) velocity in miles per hour t seconds after the start of the race. Write an equation involving one or more integrals that expresses the following statement:

N seconds after the start of the race, Evita is M miles ahead of Lar.

Your answer may involve $v(t)$ and $w(t)$.

Answer: $\frac{1}{3600}\int_0^N (v(t) - w(t))\,dt = M$

5. [10 points] The table below gives several values of a function $q(u)$ and its first and second derivatives. Assume that all of $q(u)$, $q'(u)$, and $q''(u)$ are defined and continuous for all real numbers u.

u	0	1	2	3	4	5	6
$q(u)$	30	23	19	20	24	25	24
$q'(u)$	0	-6	-2	1	3	1	-2
$q''(u)$	-9	5	4	3	2	-5	0

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

a. [2 points] Compute $\int_5^2 q''(t)\,dt$.

Solution: $\int_5^2 q''(t)\,dt = q'(2) - q'(5) = -2 - 1 = -3$.

Answer: $\int_5^2 q''(t)\,dt =$ -3

b. [2 points] Compute $\int_1^5 (-2q''(u) + 2u)\,du$.

Solution: $\int_1^5 (-2q''(u)+2u)\,du = (-2q'(5)+5^2)-(-2q'(1)+1^2) = (-2+25)-(12+1) = 10$.

Answer: $\int_1^5 (-2q''(u) + 2u)\,du =$ 10

c. [2 points] Suppose that $q(u)$ is an even function. Compute $\int_{-5}^5 q(u)\,du$.

Solution: $\int_{-5}^5 q(u)\,du = 2\int_0^5 q(u)\,du$. This cannot be computed exactly.

Answer: $\int_{-5}^5 q(u)\,du =$ **not possible**

d. [2 points] Suppose that $q(u)$ is an even function. Compute $\int_{-5}^5 \left(q'(u) + 7\right)\,du$.

Solution: $\int_{-5}^5 (q'(u)+7)\,du = (q(5) + 7\cdot 5) - (q(-5) + 7\cdot(-5))$. Since $q(5) = q(-5)$, we have $\int_{-5}^5 (q'(u)+7)\,du = q(5) - q(-5) + 7\cdot 10 = 70$.

Answer: $\int_{-5}^5 \left(q'(u) + 7\right)\,du =$ 70

e. [2 points] Compute the average value of $-5q'(u)$ on the interval $[1,4]$.

Solution: Average value $= \frac{1}{4-1}\int_1^4 -5q'(u)\,du = \frac{1}{3}\left[-5q(4) - (-5q(1))\right] = \frac{5}{3}\left[q(1) - q(4)\right] = \frac{-5}{3}$.

Answer: $\frac{-5}{3}$