The Fundamental Theorem And Interpretations

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1 The Integral Notion (and the units)

The notation of a definite integral comes from the summation notation (sigma notation). Recall that

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty}\left(\sum_{i=1}^n f(t_i)\Delta t\right).$$

So $\sum \rightsquigarrow \int$, $\sum_{i=1}^{n} \rightsquigarrow \int_a^b$, $\Delta t \rightsquigarrow \mathrm{d}t$.

The integrand function $f(t)$ has some unit (e.g., meters/sec) and so does the variable t (e.g., sec). Then the integral

$$\int_a^b f(t)\mathrm{d}t = \text{summation of (meters/sec)} \times \text{sec} = \text{summation of meters}$$

has unit meters.

2 The fundamental theorem of calculus

Theorem 2.1 (FTC)**.** *If f is continous on the interval $[a,b]$ and $f(t) = F'(t)$, then*

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

To understand FTC, note that over each subdivision $[t_{i-1}, t_i]$, we have

$$F(t_i) - F(t_{i-1}) \approx F'(t_{i-1}) \cdot (t_i - t_{i-1}) = f(t_{i-1})\Delta t$$

Therefore

$$\begin{aligned}
\sum_{i=1}^n F(t_i) - F(t_{i-1}) &\approx \sum_{i=1}^n f(t_{i-1})\Delta t \\
F(b) - F(t_{n-1}) + F(t_{n-1}) - F(t_{n-2}) + \cdots + F(t_1) - F(a) &\approx \sum_{i=1}^n f(t_{i-1})\Delta t \\
F(b) - F(a) &\approx \sum_{i=1}^n f(t_{i-1})\Delta t \\
F(b) - F(a) &= \lim_{n\to\infty}\sum_{i=1}^n f(t_{i-1})\Delta t = \int_a^b f(t)\mathrm{d}t
\end{aligned}$$

3 Interpretation (by FTC)

If $F(t)$ is a function such that $F'(t) = f(t)$, then FTC tells us that

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a).$$

So the interpretation of the integral $\int_a^b f(t)\mathrm{d}t$ is $F(b) - F(a)$, i.e., the change in F from a to b.

Example 3.1. Pollution is removed from a lake on day t at a rate of $f(t)$ kg/day. Give the meaning of $\int_5^{15} f(t)\mathrm{d}t = 4000$.

4000 kg pollution is removed from the lake from day 5 to day 15.

Example 3.2. A bungee jumper leaps off the starting platform at time $t = 0$ and rebounds once during the first 5 seconds. Let $v(t)$ be the velocity of the jumper. Then what does $\int_0^5 v(t)\mathrm{d}t$ represent in terms of the jump?

It represnets the chang in the height of jumper from start to the fifth second.

4 Questions

1. The graph of f' is given below.

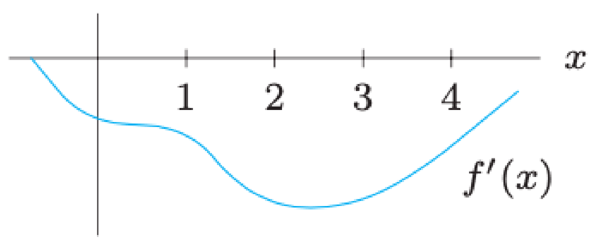

Figure 5.46: Graph of f', not f

- which is greater, $f(0)$ or $f(1)$?
- List the following in increasing order:

(1) $\dfrac{f(4) - f(2)}{2}$

(2) $f(3) - f(2)$

(3) $f(4) - f(3)$

2. Use the FTC to find integrals of $\int_a^b F'(t)\mathrm{d}t$ where

   (a) $F(t) = t^2; a = 1, b = 3$

   (b) $F(t) = \ln(t); a = 1, b = 5$

   (c) $F(t) = \tan(t); a = 0, b = \pi.$

3. Evaluate the expressions using table below. Give exact values if possible; otherwise, make the best possible estimates using left-hand Riemann sums.

t	0.0	0.1	0.2	0.3	0.4	0.5
$f(t)$	0.3	0.2	0.2	0.3	0.4	0.5
$g(t)$	2.0	2.9	5.1	5.1	3.9	0.8

   (a) $\int_0^{0.5} f(t)\mathrm{d}t$

   (b) $\int_{0.2}^{0.5} g'(t)\mathrm{d}t$

   (c) $\int_0^{0.3} g(f(t))\mathrm{d}t$

4. [14 points] One of the ways Captain Christina likes to relax in her retirement is to go for long walks around her neighborhood. She has noticed that early every Tuesday morning, a truck delivers butter to a local bakery famous for its cookie dough. Consider the following functions:

- Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of butter.
- Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter.
- Let $u(t)$ be the instantaneous rate, in pounds per hour, at which butter is being unloaded t hours after 4 am.

Assume that C, K, and u are invertible and differentiable.

a. [2 points] Interpret $K(C^{-1}(10)) = 20$ in the context of this problem.
Use a complete sentence and include units.

b. [3 points] Interpret $\int_5^{12} K'(b)\, db = 40$ in the context of this problem.
Use a complete sentence and include units.

c. [3 points] Give a single mathematical equality involving the derivative of C which supports the following claim:
It costs the bakery approximately \$0.70 less to buy 14.8 pounds of butter than to buy 15 pounds of butter.

Answer: ____________________

d. [3 points] Give a single mathematical equality which expresses the following claim:
The number of pounds of butter unloaded between 5 and 8 am is twice as many as the bakery needs to make 5000 cups of cookie dough.

Answer: ____________________

e. [3 points] Assume that $u(t) > 0$ and $u'(t) < 0$ for $0 \le t \le 4$ and that $u(2) = 800$.
Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-IV.

I. 0 II. 800 III. $\int_1^2 u(t)\, dt$ IV. $\int_2^3 u(t)\, dt$

______ $<$ ______ $<$ ______ $<$ ______

8. [11 points] The energy, in megajoules (MJ), produced by a wind turbine depends on the speed of the wind. In particular, suppose $P(s)$ is the power, in megajoules per hour (MJ/h), produced by the turbine when the speed of the wind is s kilometers per hour (km/h). Also suppose that $W(t)$ gives the wind speed, in km/h, at the turbine's location t hours after noon on a typical day.

Assume that $P(s)$ is invertible, and that both $P(s)$ and $W(t)$ are differentiable.

a. [2 points] Give a practical interpretation of the equation $P(W(0)) = 8$.

b. [3 points] Give a practical interpretation of the equation $\int_0^5 P(W(t))\, dt = 46$.

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(4) = 21$$

From 4 pm to 4:10 pm, . . .

d. [3 points] Circle the one statement below that is best supported by the equation

$$(P^{-1})'(13) = 2.9.$$

i. If the turbine is producing 13 MJ/h of power, the wind speed must increase by approximately 2.9 km/h to produce an additional MJ/h of power.

ii. If the wind is blowing at 13 km/h and increases to 14 km/h, the power produced by the turbine will increase by about 2.9 MJ/h.

iii. If the wind speed is 13 km/h, the power generation of the turbine will increase by one MJ/h if the wind speed increases to about 15.9 km/h.

iv. When the turbine is generating 13 MJ/h of power, an increase of one km/h in wind speed will produce approximately 2.9 MJ/h more power.

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