The Definite Integral

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April 09, 2020

1 Sigma Notation

Suppose that $f(t)$ is a continuous function $a \leqslant t \leqslant b$. We divide the integral from a to b into n *equal* subdivisions, and we call the width of an individual subdivision Δt. Then $\Delta t = \dfrac{b-a}{n}$.

Let $t_0, \ldots, t_n$ be endpoints of the subdivisions. Then

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$$

$$\text{Left-hand sum} = f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t$$

Here $\sum$ is a capital sigma, or Greek letter "S", it means the "sum" of certain terms.

Write out following sums explicitly

- $\sum\limits_{i=1}^{3} i^2$
- $\sum\limits_{i=1}^{4} 2^i$
- $\sum\limits_{i=1}^{3} \dfrac{1}{i} \cdot t$

2 Taking the Limit to Obtain the Definite Integral

Now we let $n \to \infty$. If f is continuous on the interval $[a, b]$, then the limits of the left- and right-hand sum exits and are equal. The *definite integral* is the limit of the sums.

Definition 2.1. Suppose that f is continuous for $a \leqslant t \leqslant b$. The *definite integral* of f from a to b, written

$$\int_a^b f(t)\mathrm{d}t$$

is the limit of the left-hand or right-hand sums with n subdivisions of $[a, b]$ as n gets large, i.e.,

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty}(\text{Left-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(t_i)\Delta t\right)$$

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty}(\text{Right-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\Delta t\right)$$

Some new terms:

- f is called the *integrand*;
- a, b are called the *limits of integration*;
- The left-hand sum and right-hand sum are *Riemann sums*.

Example 2.2. Use the table to estimate $\int_0^{40} f(x)\mathrm{d}x$. What values of n and Δx did you use?

x	0	10	20	30	40
$f(x)$	350	410	435	450	460

2.1 Definite integral as an area

How do we compute a definite integral? Just like what we did from last class, the definite integral is actually the area under the curve.

Proposition 2.3. *When $f(x) \geqslant 0$ and $a < b$, area under the graph of f and above x-axis between a and b is*

$$\int_a^b f(t)\mathrm{d}t$$

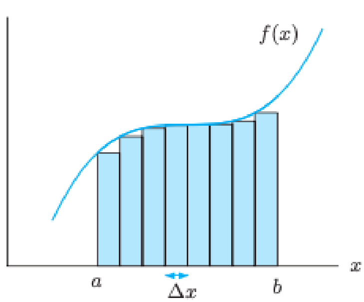


Figure 5.23: Area of rectangles approximating the area under the curve

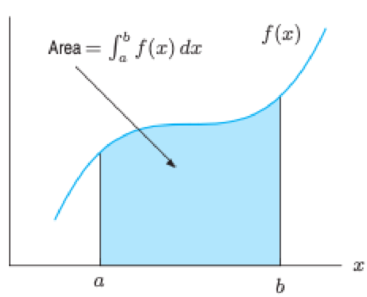


Figure 5.24: The definite integral $\int_a^b f(x)\,dx$

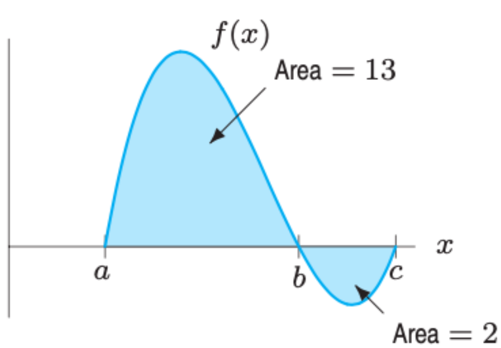
Example 2.4. Consider the integral $\int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x$

When $f(x)$ is not always positive,

Proposition 2.5. *When $f(x)$ is positive for some x and negative for others, and $a < b$, the definite integral $\int_a^b f(x)\mathrm{d}x$ is the sum of areas above the x-axis, counted positively, and the area under x-axis, counted negatively.*

Example 2.6. Use the figure below to find the values of

1. $\displaystyle\int_a^b f(x)\mathrm{d}x$
2. $\displaystyle\int_b^c f(x)\mathrm{d}x$
3. $\displaystyle\int_a^c f(x)\mathrm{d}x$
4. $\displaystyle\int_a^c |f(x)|\mathrm{d}x$



3 Reimann Sums

A general *Riemann sum* for f on the interval $[a, b]$ is a sum of the form

$$\sum_{i=1}^{n} f(c_i)\Delta t_i$$

where $a = t_0 < t_1 < \cdots < t_n = b$, and, for $i = 1, \ldots, n$, $\Delta t_i = t_i - t_{i-1}$ and $t_{i-1} \leqslant c_i \leqslant t_i$.

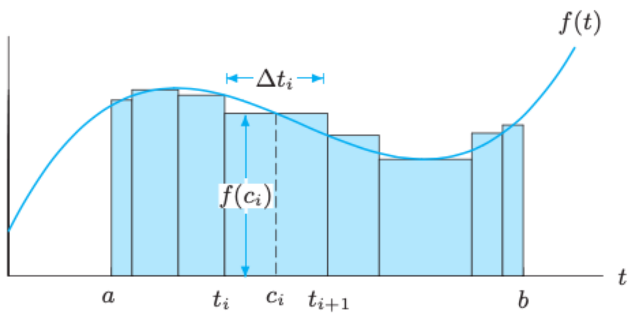


Figure 5.28: A general Riemann sum approximating $\int_a^b f(t)\,dt$

There is a very useful online calculator to help you understand Riemann sum

https://mathworld.wolfram.com/RiemannSum.html

1. [11 points] At a recent UM football game, a football scientist was measuring the excitement density, $E(x)$, in cheers per foot, in a one hundred foot row of the football stadium where x is the distance in feet from the beginning of the row. He took measurements every twenty feet and the data is recorded in this table.

x	0	20	40	60	80	100
$E(x)$	30	24	19	16	13	7

Assume for this problem that $E(x)$ is a decreasing function for $0 \leq x \leq 100$.

a. [6 points] Write a right sum and a left sum which approximate the total cheers in the row. Be sure to write all of the terms for each sum.

b. [2 points] Indicate whether the right and left sums are overestimates or underestimates for the total number of cheers in the row.

The right sum is an **overestimate** **underestimate**

The left sum is an **overestimate** **underestimate**

2. [8 points] Due to an accident, an oil pipeline is leaking.
Let $p(t)$ be the rate (in gallons/hour) at which the pipeline leaks oil t hours after the accident.
Assume that $p(t)$ is a strictly decreasing, differentiable function for $0 \leq t \leq 24$.
Engineers make the following measurements of $p(t)$.

t	0	6	12	18	24
$p(t)$	97	86	79	61	49

b. [3 points] Based on the data provided, write the right Riemann sum that best approximates the total amount of oil (in gallons) that leaked from the pipeline in the first 24 hours after the accident. *Be sure to carefully write out all of the terms in the sum.*

c. [1 point] Indicate whether the right sum above is an overestimate or an underestimate for the total amount of oil leaked. If there is not enough information to make this determination, circle "not enough information". You do not need to explain your answer.

Answer: The right sum is an (circle one):

overestimate underestimate not enough information

4. [10 points] Gabe the mouse is swimming alone in a very large puddle of water. He keeps track of his swimming time by logging his velocity at various points in time. Gabe starts at a point on the edge of the puddle and swims in a straight line with increasing speed. A table of Gabe's velocity $V(t)$, in feet per second, t seconds after he begins swimming is given below.

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$V(t)$	0	0.3	0.4	0.45	0.9	1.2	1.8	2.4	2.7	2.9	3	3.2	3.5

b. [3 points] Estimate $\int_1^{5.5} V(t)\,dt$ by using a right-hand Riemann sum with 3 equal subdivisions. Make sure to write down all terms in your sum.

Answer: ______________________

c. [1 point] Is your estimate from above an overestimate or an underestimate of the exact value of $\int_1^{5.5} V(t)\,dt$? Circle your answer.

OVERESTIMATE UNDERESTIMATE NOT ENOUGH INFORMATION