

How do we measure distance traveled

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1 Velocity Data

A car is moving with increasing velocity.

The velocities of every two seconds are shown below.

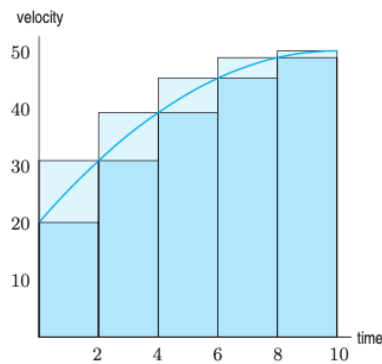
Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

To get an underestimation:

$$20 \cdot 2 + 30 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 = 360 \text{ feet}$$

To get an overestimation:

$$30 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 + 50 \cdot 2 = 420 \text{ feet}$$



Therefore the the distance traveled is between 360 feet and 420 feet.

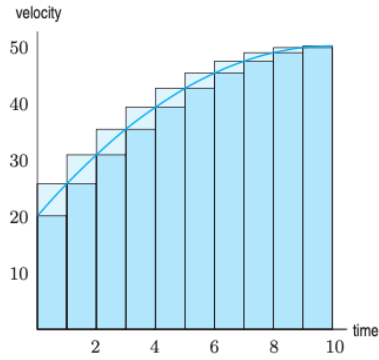
The velocities of every one second are shown below.

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	34	38	41	44	46	48	49	50

Using these data, we can find new estimates:

$$\text{underestimateion} = 20 \cdot 1 + 26 \cdot 1 + 30 \cdot 1 + 34 \cdot 1 + 38 \cdot 1 + 41 \cdot 1 + 44 \cdot 1 + 46 \cdot 1 + 48 \cdot 1 + 49 \cdot 1 = 376 \text{ feet}$$

$$\text{overestimation} = 26 \cdot 1 + 30 \cdot 1 + 34 \cdot 1 + 38 \cdot 1 + 41 \cdot 1 + 44 \cdot 1 + 46 \cdot 1 + 48 \cdot 1 + 49 \cdot 1 + 50 \cdot 1 = 406 \text{ feet}$$



1.1 Area under the curve

What happens if we split the interval $[0, 10]$ into more and more subdivisions?

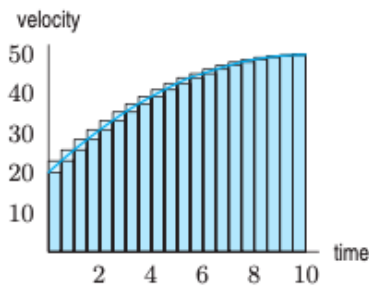


Figure 5.3: Velocity measured every $1/2$ second

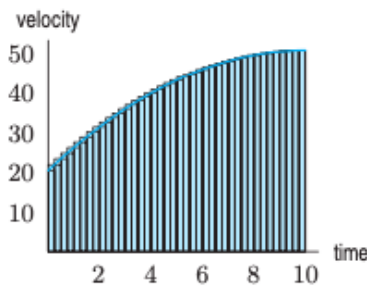


Figure 5.4: Velocity measured every $1/4$ second

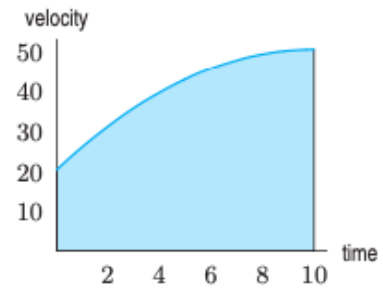


Figure 5.5: Distance traveled is area under curve

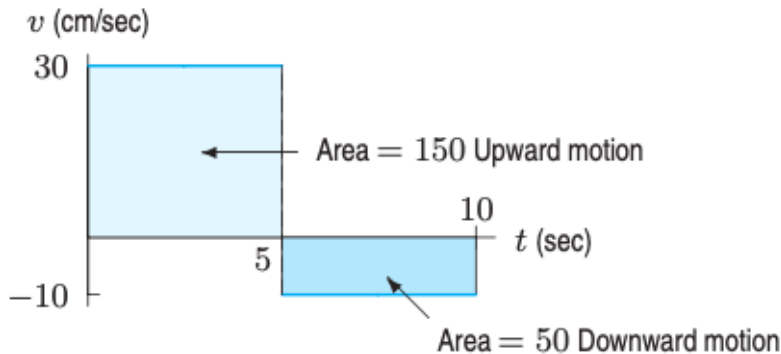
The total sum is approaching the area under the curve!

Hence we have following conclusion:

If the velocity is positive, the total distance traveled is the area under the velocity curve.

1.2 Negative velocity and change in position

A particle moves along the y -axis with velocity 30 cm/sec for 5 seconds and velocity -10 cm/sec for the next 5 seconds. Positive velocity indicates upward motion; negative velocity represents downward motion. What does the sum $30 \cdot 5 + (-10) \cdot 5$ represent?



2 More Discussions

2.1 Left and Right sums

Suppose that $f(t)$ is a continuous function $a \leq t \leq b$. We divide the interval from a to b into n equal subdivisions, and we call the width of an individual subdivision Δt . Then $\Delta t = \frac{b-a}{n}$.

Let t_0, \dots, t_n be endpoints of the subdivisions. Then

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t$$

$$\text{Left-hand sum} = f(t_0)\Delta t + f(t_1)\Delta t + \dots + f(t_{n-1})\Delta t$$

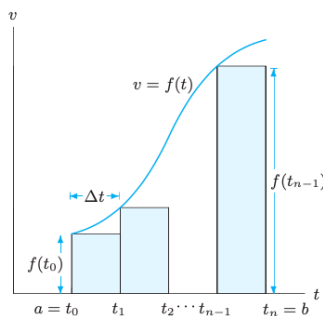


Figure 5.8: Left-hand sums

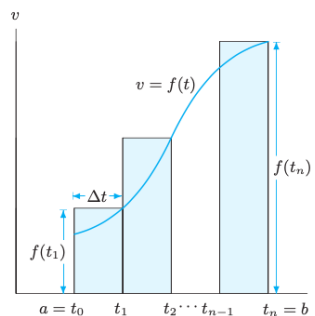


Figure 5.9: Right-hand sums

2.2 Accuracy of Estimates

For either increasing or decreasing velocity functions, the exact value of the distance traveled lies somewhere between the two estimates. Thus, the accuracy of our estimate depends on how close these two sums are.

For a function which is increasing throughout or decreasing throughout the interval $[a, b]$:

$$\begin{aligned} |\text{Difference between upper and lower estimations}| &= |\text{right-hand sum} - \text{left-hand sum}| \\ &= |(f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t) \\ &\quad - (f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t)| \\ &= |f(t_n)\Delta t - f(t_0)\Delta t| \\ &= |f(b) - f(a)| \cdot \Delta t \\ &= |\text{difference between } f(a) \text{ and } f(b)| \cdot \Delta t \end{aligned}$$

3 Questions

1. A car comes to a stop five seconds after the driver applies the brakes. While the brakes are on, the velocities in the table are recorded.

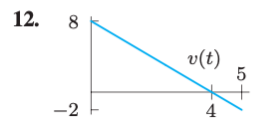
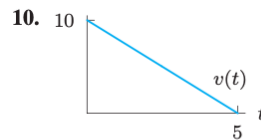
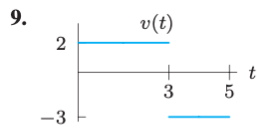
Time since brakes applied (sec)	0	1	2	3	4	5
Velocity (ft/sec)	88	60	40	25	10	0

- (a) Give lower and upper estimates of the distance the car traveled after the brakes were applied.
(b) On a sketch of velocity against time, show the lower and upper estimates of part (a).
(c) Find the difference between the estimates.

2. Roger runs a marathon. His friend Jeff rides behind him on a bicycle and clocks his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:

Times since start (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

- (a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the first half hour.
- (b) Give upper and lower estimates for the distance Roger ran in total during the entire hour and a half.
- (c) How often would Jeff have needed to measure Roger's speed in order to find lower and upper estimates within 0.1 mile of the actual distance he ran?
3. Graphs below show the velocity, in cm/sec, of a particle moving along a number line. Positive velocities represent movement to the right; negative velocities to the left. Compute the change in position between times $t = 0$ and $t = 5$ seconds.



4. Find the difference between the upper and lower estimates of the distance traveled at velocity $f(t)$ on the interval $a \leq t \leq b$ for n subdivisions.
- (a) $f(t) = 5t + 8, a = 1, b = 3, n = 100$.
 - (b) $f(t) = 25 - t^2, a = 1, b = 4, n = 500$.
 - (c) $f(t) = \sin(t), a = 0, b = \frac{\pi}{2}, 100$.

5. Two cars start at the same time and travel in the same direction along a straight road. Figure below gives the velocity v of each car as a function of time t . Which car:

- 1. Attains the larger maximum velocity?
- 2. Stops first?
- 3. Travels farther?

