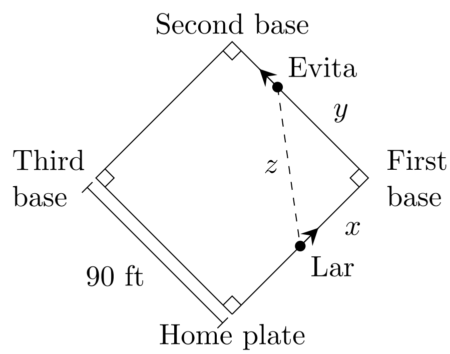
5. [9 points]
During the annual Srebmun Foyoj kickball game, Lar Getni kicks the ball and runs from home plate to first base, while Evita Vired runs from first base to second base.

Let x be the distance between Lar and first base, y be the distance between Evita and first base, and z be the distance between Lar and Evita, as shown in the diagram on the right. Note that the bases are arranged in a square and that the distance between consecutive bases is 90 feet.



At the moment when Lar is halfway from home plate to first base, Evita is two thirds of the way from first base to second base. At this moment, Lar is running at a speed of 32 ft/s, and Evita is running at a speed of 36 ft/s. The questions below all refer to this moment.

Throughout this problem, remember to show your work clearly, and include units in your answers.

a. [5 points] At the moment when Lar is halfway to first base, at what rate is the distance between Lar and Evita changing? Is the distance increasing or decreasing?

Solution: We need to find $\frac{dz}{dt}$ at the moment when Lar is halfway to first base. By the Pythagorean Theorem, we have $z^2 = x^2 + y^2$. Differentiating both sides of this equation with respect to time, we find that $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$.

At the moment in question, we have $x = 45$ ft, $y = 60$ ft, $z = \sqrt{45^2 + 60^2} = 75$ ft, $\frac{dx}{dt} = -32$ ft/s, and $\frac{dy}{dt} = 36$ ft/s. So at this moment,

$$\frac{dz}{dt} = \frac{2(45)(-32) + 2(60)(36)}{2(75)} = 9.6 \text{ ft/s}.$$

Answer: The distance is (circle one) $\boxed{\text{INCREASING}}$ DECREASING

at a rate of 9.6 **ft/s** .

b. [4 points] At the moment when Lar is halfway to first base, at what rate is the area of the right triangle formed by Lar, Evita, and first base changing? Is the area increasing or decreasing?

Solution: The area of the triangle is given by $A = \frac{xy}{2}$. Differentiating both sides of this equation with respect to time, we find that $\frac{dA}{dt} = \frac{x\frac{dy}{dt} + y\frac{dx}{dt}}{2}$.

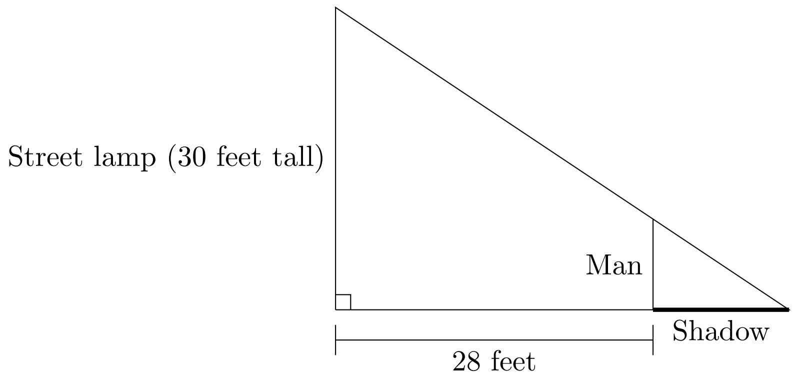
At the moment when Lar is halfway to first base, we therefore have

$$\frac{dA}{dt} = \frac{(45)(36) + (60)(-32)}{2} = -150 \text{ ft}^2\text{/s}.$$

Answer: The area is (circle one) INCREASING $\boxed{\text{DECREASING}}$

at a rate of 150 $\mathbf{ft^2/s}$.

3. [8 points] A man, who is 28 feet away from a 30 foot tall street lamp, is sinking into quicksand. (*See diagram below.*) At the moment when 6 feet of him are above the ground, his height above the ground is shrinking at a rate of 2 feet/second.



Throughout this problem, remember to show your work clearly, and include units in your answers.

a. [3 points] How long will the man's shadow (shown in bold in the diagram above) be at the moment when 6 feet of him are above the ground?

Solution: Let s be the length of the shadow. Noticing that the larger and smaller triangles in the picture are similar triangles, we have

$$\begin{aligned} \frac{30}{28+s} &= \frac{6}{s} \\ 30s &= 168 + 6s \\ 24s &= 168 \\ s &= 7. \end{aligned}$$

So the length of the shadow is 7 feet at that moment.

Answer: **7 feet**

b. [5 points] At what rate is the length of the man's shadow changing at the moment 6 feet of him are above the ground? Is his shadow growing or shrinking at that moment?

Solution: Let h be the height of the man above the ground, and let s be the length of his shadow. Using similar triangles as above, we have $\dfrac{30}{28+s} = \dfrac{h}{s}$ so $30s = 28h + hs$.

Taking derivatives with respect to time t, we find $30\dfrac{ds}{dt} = 28\dfrac{dh}{dt} + h\dfrac{ds}{dt} + s\dfrac{dh}{dt}$.

So at the moment when $h = 6$, we have

$$\begin{aligned} 30\left.\frac{ds}{dt}\right|_{h=6} &= 28(-2) + 6\left.\frac{ds}{dt}\right|_{h=6} + 7(-2) \\ 24\left.\frac{ds}{dt}\right|_{h=6} &= -70 \\ \left.\frac{ds}{dt}\right|_{h=6} &= \frac{-70}{24} = -\frac{35}{12} \approx -2.917 \end{aligned}$$

So at that moment, the shadow is shrinking at a rate of about 2.917 feet/second.

Answer: The man's shadow is (circle one) GROWING $\boxed{\text{SHRINKING}}$

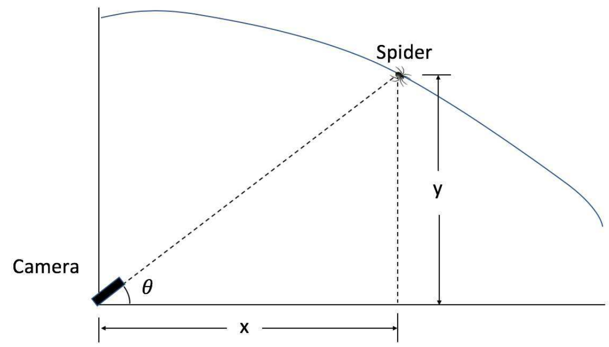
at a rate of $\frac{35}{12}$ **(about 2.917) feet/second** .

4. [7 points] Casey is making a documentary about the wildlife that lives in a local cave. She found a spider of a new species climbing down along the ceiling of the cave (as shown in the diagram below). Here

- x is the spider's distance to the right, in ft, of the camera
- y is the height, in ft, of the spider from the ground
- θ is the angle, in radians, made by the ground and the line joining Casey's camera and the spider.

The camera is following the spider as it walks along the ceiling of the cave. **Find the rate at which the angle θ is changing** when the following conditions hold:

- The spider is 10 ft above the ground.
- The spider's distance to the right of the camera is increasing at 0.4 feet per second.
- The spider's height is decreasing at a rate of 0.2 feet per second.
- The angle $\theta = \frac{\pi}{6}$ radians.



Use the equation

$$\tan(\theta) = \frac{y}{x}.$$

satisfied by the variables x, y and θ to find your answer. Include units. Show all your work.

Solution: Taking derivatives with respect to time we get

$$\frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}.$$

Solving for $\frac{d\theta}{dt}$ we get

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}\cos^2(\theta)$$

We are given $y = 10$, $\frac{dx}{dt} = 0.4$, $\frac{dy}{dt} = -0.2$ and $\theta = \frac{\pi}{6}$ then $\tan(\theta) = \frac{y}{x}$ yields $\frac{10}{x} = \frac{1}{\sqrt{3}}$ or $x = 10\sqrt{3}$. Using these values into this equation, we get

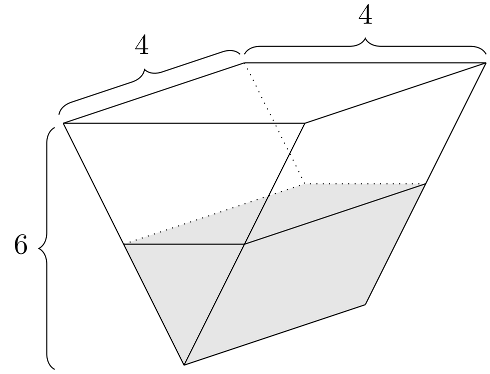
$$\frac{d\theta}{dt} = \frac{10\sqrt{3}(-0.2) - 10(0.4)}{(10\sqrt{3})^2}\cos^2\left(\frac{\pi}{6}\right) = \frac{-2\sqrt{3}-4}{300}\left(\frac{3}{4}\right) = \frac{-\sqrt{3}-2}{200}$$

Answer: $\frac{-\sqrt{3}-2}{200}$ radians per second.

4. [10 points]

a. [5 points]

Sam is pouring concrete into a hole in the shape of a triangular prism. The hole is 4 meters wide, 4 meters long, and 6 meters deep at its deepest point. A partially filled hole with the correct dimensions is shown to the right.



Sam is looking down into the hole and observes that the rectangular top surface of the concrete is growing at a rate of 0.8 meters squared per minute. Find the rate at which the depth of the concrete is growing. *Include units.*

Solution: Let w denote the width of the surface rectangle and h denote the depth of the concrete. By similar triangles, $w = \frac{2}{3}h$. Then the area, $A(h)$, is

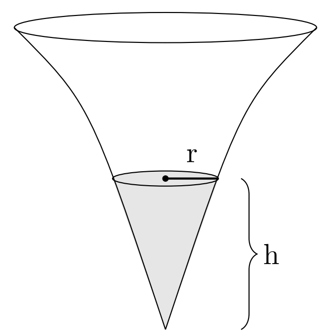
$$A(h) = 4\left(\frac{2}{3}h\right) = \frac{8}{3}h.$$

Taking the derivative with respect to time we have $\frac{dA}{dt} = \frac{8}{3}\frac{dh}{dt}$, so $\frac{dh}{dt} = 0.8\left(\frac{3}{8}\right) = 0.3.$

Answer: 0.3 **meters per minute**

b. [5 points]

Donna is pouring concrete into a different hole, which is in the shape of a horn as shown to the right. When the concrete has been poured to a depth of h meters and its surface has radius r, the volume of the poured concrete is given by



$$V = \frac{\pi}{7}r^2h.$$

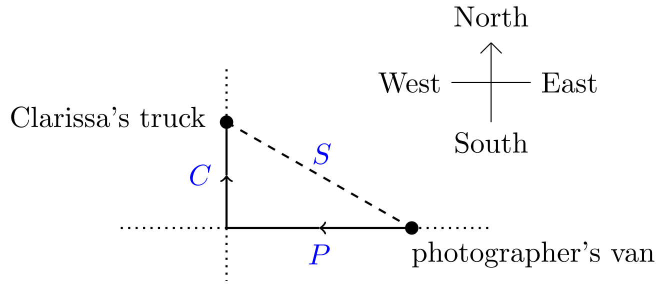
When the depth of the concrete that has been poured is 0.8 meters, the radius of its surface is 0.5 meters, the radius is growing at a rate of 5 meters per hour, and the volume is growing at a rate of 2 cubic meters per hour. How fast is the depth changing? *Include units.*

Solution:

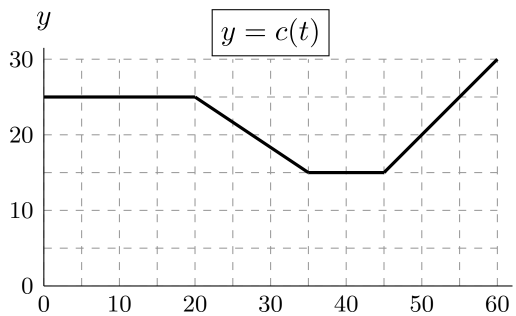
$$\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{7}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)\\
2 &= \frac{\pi}{7}\left(2(0.5)(0.8)(5) + (.5)^2\frac{dh}{dt}\right)\\
\frac{14}{\pi} &= 4 + 0.25\frac{dh}{dt}\\
\frac{14}{\pi} - 4 &= 0.25\frac{dh}{dt}\\
4\left(\frac{14}{\pi} - 4\right) &= \frac{dh}{dt}
\end{aligned}$$

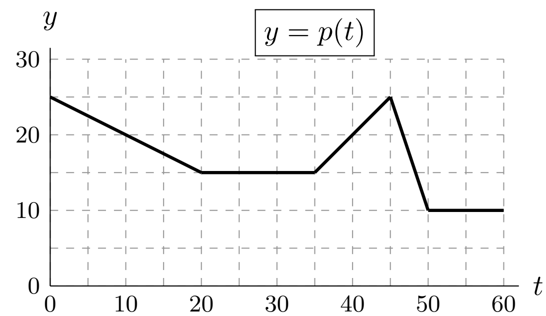
Answer: $4\left(\frac{14}{\pi} - 4\right) \approx 1.825$ **meters per hour**

7. [10 points] Clarissa is driving a truck of showpigs, heading due north away from an intersection, while her photographer's van is due east of the intersection and driving due west, as shown to the right.



For time t measured in seconds after noon, let $c(t)$ be the speed of Clarissa's truck, and let $p(t)$ be the speed of the photographer's van, both measured in meters per second (m/s). Graphs of $c(t)$ and $p(t)$ are shown below.





At $t = 20$, Clarissa is 1000 meters from the intersection, and her photographer's van is 2400 meters from the intersection. Throughout this problem, be sure your work is clear.

a. [5 points] At $t = 20$, is the distance between the two vehicles increasing or decreasing? At what rate?

Solution: We can label the triangle as in the above picture. By the Pythagorean Theorem, $C^2 + P^2 = S^2$. Taking the derivative of both sides, we see that

$$2C\frac{dC}{dt} + 2P\frac{dP}{dt} = 2S\frac{dS}{dt}.$$

At time $t = 20$, we have $C = 1000$, $P = 2400$, and $S = \sqrt{C^2 + P^2} = 2600$. Since Clarissa is moving *away* from the intersection, $\frac{dC}{dt} = c(20) = 25$ and since the photographer is moving *towards* it, $\frac{dP}{dt} = -p(20) = -15$. Plugging in this information, we find that $\frac{dS}{dt} \approx -4.23077$.

Answer: INCREASING DECREASING at a rate of: ≈ 4.23077 m/s

b. [2 points] How far from the intersection is the photographer's van at $t = 35$?

Solution:

$$P(35) = 2400 - \int_{20}^{35} p(t)dt = 2400 - 225 = 2175$$

Answer: 2175 meters

c. [3 points] What is the distance, in meters, between the two vehicles at $t = 35$?

Solution:

$$C(35) = 1000 + \int_{20}^{35} c(t)dt = 1000 + 300 = 1300$$

so using our answer from part (b), $S(35) = \sqrt{(2175)^2 + (1300)^2} \approx 2533.90$.

Answer: 2533.9 meters