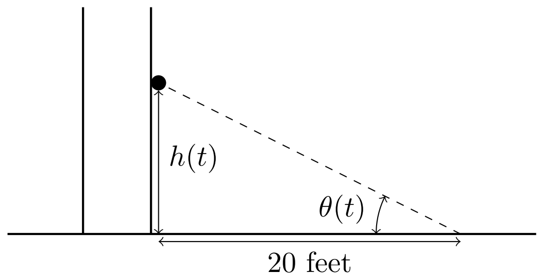
6. [12 points] Walking through Nichols Arboretum, you see a squirrel running down the trunk of a tree. The trunk of the tree is perfectly straight and makes a right angle with the ground. You stop 20 feet away from the tree and lie down on the ground to watch the squirrel. Suppose $h(t)$ is the distance in feet between the squirrel and the ground, and $\theta(t)$ is the angle in radians between the ground and your line of sight to the squirrel, with t being the amount of time in seconds since you stopped to watch the squirrel.



a. [3 points] Write an equation relating $h(t)$ and $\theta(t)$. (Hint: Use the tangent function.)

Solution:

$$\tan\theta(t) = \frac{h(t)}{20}$$

b. [5 points] If $\theta(t)$ is decreasing at 1/5 of a radian per second when $\theta(t) = \pi/3$, how fast is the squirrel moving at that time?

Solution: Differentiate with respect to t:

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

Plug in $\theta'(t) = -1/5$ and $\theta(t) = \pi/3$:

$$\frac{1}{\cos^2(\pi/3)}(-1/5) = \frac{h'(t)}{20}$$

Solve to get $h'(t) = -16$ so the squirrel is moving at -16 feet per second.

c. [4 points] For the last second before the squirrel reaches the ground, it is moving at a constant speed of 20 feet per second. Suppose $\theta'(t) = -3/4$ at some point during this last second. How high is the squirrel at this time?

Solution: Start with

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

and plug in $h'(t) = -20$ and $\theta'(t) = -3/4$:

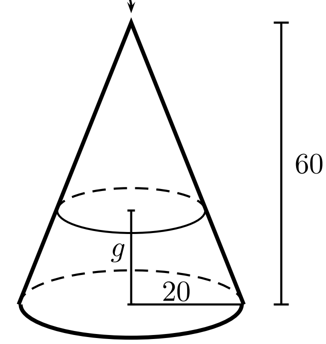
$$\frac{1}{\cos^2\theta(t)}(-3/4) = \frac{-20}{20}$$

This gives us that $\cos^2\theta(t) = 3/4$. Then $\cos\theta(t) = \sqrt{3}/2$ (positive because $0 < \theta < \pi/2$), and $\theta(t) = \arccos(\sqrt{3}/2) = \pi/6$. Finally, we use the equation from part (a) to get

$$h(t) = 20\tan(\pi/6) = 20/\sqrt{3} \approx 11.547$$

so the squirrel is at a height of about 11.547 feet.

4. [12 points]
Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.



In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \frac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

Solution: The volume of the solid is given by $V = \frac{1}{3}\pi(20)^2 60 - \frac{1}{3}\pi r^2(60-g)$ where r is the radius of the cross-section at height g. We want to rewrite r in terms of g. Using similar triangles we find the equation

$$\frac{r}{20} = \frac{60-g}{60},$$

which implies $r = \frac{60-g}{3}$. Therefore, $V = 8000\pi - \frac{\pi}{27}(60-g)^3$. So, to find the appropriate g we need to solve $8000\pi - \frac{\pi}{27}(60-g)^3 =$20,000. Solving, we get

$$(60-g)^3 = \frac{27}{\pi}(8000\pi - 20,000), \qquad (2)$$

which implies $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)} \approx 24.67$. The chocolate is approximately 24.67 mm deep when he has poured a total of 20,000 mm^3 of chocolate into the mould.

Answer: $g \approx$ **24.67**

b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Solution: We want to find $\frac{dg}{dt}$ when $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}$ (from part (a)) if $\frac{dV}{dt} = 5000$ at this time. Differentiating our formula (2) from part (a) with respect to t, we have

$$\frac{dV}{dt} = \frac{\pi}{9}(60-g)^2\frac{dg}{dt}.$$

Substituting $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}$ and $\frac{dV}{dt} = 5000$ into this equation, we find

$$5000 = \frac{\pi}{9}\left(60 - \left[60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}\right]\right)^2 \frac{dg}{dt} \quad \text{so}$$

$$\frac{dg}{dt} = \frac{5000 \cdot 9}{\pi}\left(\frac{27}{\pi}(8000\pi - 20,000)\right)^{-2/3} \approx 11.47.$$

(Using our approximation $g \approx 24.67$ instead gives us $\frac{dg}{dt} \approx 11.48$.)
So the depth of the chocolate is increasing at an instantaneous rate of about 11.47 mm/sec at that moment.

Answer: **11.47 mm/ sec.**