Rates and Related Rates

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1 Rates

Derivatives represents rates of change.

Let start with an example

Example 1.1. A spherical snowball is melting. Its radius decreases at a constant rate of 2 cm per minute from an initial value of 70 cm. How fast is the volume decreasing half an hour later?

The radius r starts at 70 cm and decreases at 2 cm/min. At time t minures since the start, we have

$$r = \_\_\_\_\_ \quad \text{cm}$$

The volume of a ball is given by $V = \frac{4}{3}\pi r^3$. Hence

$$V = \_\_\_\_\_ \quad \text{cm}^3$$

The rate at which the volume is changing at time t is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \_\_\_\_\_ \quad \text{cm}^3/\text{min}$$

After half an hour ($t = 30$ mins), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{t=30} = \_\_\_\_\_ \quad \text{cm}^3/\text{min}$$

Thus, the rate at which the volume is increasing is _____half an hour later.

Alternative solution:

2 Related Rates

Example 2.1. A spherical snowball is melting. Its radius decreases at a instant rate of 3 cm per minute when the radius is 20cm. How fast is the volume decreasing at that instant?

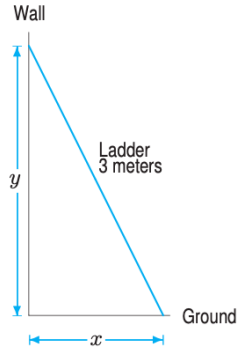
Similiarily, we have

$$V = \_\_\_\_\_$$
$$V' = \_\_\_\_\_$$

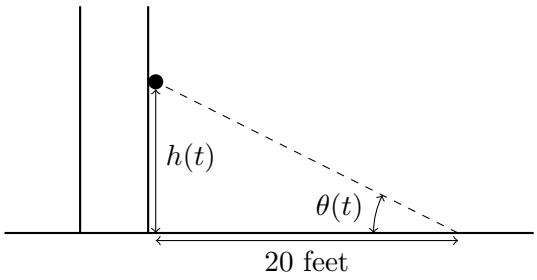
Thus, the rate at which the volume is increasing is _____at that instant.

One more example,

Example 2.2. A 3-meter ladder stands against a high wall. The foot of the ladder moves outward at a constant speed of 0.1 meter/sec. When the foot is 1 meter from the wall, how fast is the top of the ladder falling? What about when the foot is 2 meters from the wall?



6. [12 points] Walking through Nichols Arboretum, you see a squirrel running down the trunk of a tree. The trunk of the tree is perfectly straight and makes a right angle with the ground. You stop 20 feet away from the tree and lie down on the ground to watch the squirrel. Suppose $h(t)$ is the distance in feet between the squirrel and the ground, and $\theta(t)$ is the angle in radians between the ground and your line of sight to the squirrel, with t being the amount of time in seconds since you stopped to watch the squirrel.



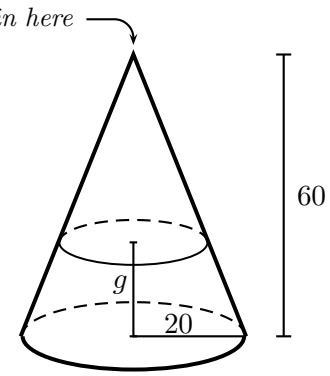
a. [3 points] Write an equation relating $h(t)$ and $\theta(t)$. (Hint: Use the tangent function.)

b. [5 points] If $\theta(t)$ is decreasing at $1/5$ of a radian per second when $\theta(t) = \pi/3$, how fast is the squirrel moving at that time?

c. [4 points] For the last second before the squirrel reaches the ground, it is moving at a constant speed of 20 feet per second. Suppose $\theta'(t) = -3/4$ at some point during this last second. How high is the squirrel at this time?

4. [12 points]

Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.



In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \tfrac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

Answer: $g \approx$ _____

b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Answer: _____