Applications to Marginality

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1 Definition

- The *cost function* $C(q)$ gives the total cost of producing a quantity q of some good.
- The *revenue function* $R(q)$ gives the total revenue received by by a firm from selling a quantity q of some good. Usually the revenue is given by the formula

$$\text{Revenue} = \text{Price} \times \text{Quantity}, \qquad R(q) = p \times q$$

- The *profit function* $\pi(q)$ is given by

$$\text{Profit} = \text{Revenue} - \text{Cost}, \qquad \pi(q) = R(q) - C(q)$$

2 Marginal Analysis

In real life, the quantity may not be continuous (e.g., q may be the number of cows, or number of people in our section), that's why we use the word "marginal analysis" instead of optimizations.

In a marginal analysis,

- The derivative of the cost function, $C'(q)$, is called "marginal costs", also denoted by MC.
- The derivative of the revenue function, $R'(q)$, is called "marginal revenues", also denoted by MR.

The word "marginal (cost/revenue)" comes from following: say some airline company plans on running 100 flights, which costs $C(100)$. When running one more flight, the cost goes to $C(101)$. Therefore additional cost "at the margin" is $C(101) - C(100)$. Note that

$$C(101) - C(100) = \frac{C(101) - C(100)}{101 - 100} \approx C'(100)$$

So "marginal cost" is approximating the derivative function.

Since maximum/minimum profit usually occurs at points where $\pi' = 0$, we also say that

> The maximum/minimum profit can occur when marginal cost = marginal revenue($MC = MR$).

How do we use "marginal analysis"?

Example 2.1. The revenue from selling q items is $R(q) = 500q - q^2$, and the total cost is $C(q) = 150 + 10q$. Write a function that gives the total profit earned, and find the quantity which maximizes the profit.

The total profit $\pi(q) = R(q) - C(q) = 500q - q^2 - (150 + 10q) = -q^2 + 490q - 150$.

We are looking for q that gives marginal revenue = marginal cost:

$$MR = R'(q) = 500 - 2q$$
$$MC = C'(q) = 10$$

So $500 - 2q = 10 \Rightarrow q = 245$. In order to determine whether $q = 245$ is a local maximum or minimum. We look at the second derivative $\pi''(q) = -2 < 0$. So $q = 245$ is a local maximum. Since this is the only critical point in our domain, it is also the global maximum.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \le b \le 100 \\ 4b + 200 & \text{for } 100 < b \le 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: ______________________________

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: ______________________________

For $0 \le b \le 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: ______________________________

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Answer: ______________________________

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Answer: bunches of kale: ____________________ and max profit: ____________________

5. [8 points] Reggie is starting a fruit punch company. He has determined that the total cost, in dollars, for him to produce q gallons of fruit punch can be modeled by

$$C(q) = 100 + q + 25e^{q/100}.$$

Reggie can sell up to 100 gallons to Chris at a price of \$4 per gallon, and he can sell the rest to Alice at a price of \$3 per gallon. Assume that Reggie sells all of the fruit punch that he produces.

Note: Assume that the quantities of fruit punch produced and sold do not have to be whole numbers of gallons. (For example, Reggie could produce exactly $50\sqrt{2}$ gallons of fruit punch and sell all of these to Chris, who would pay a total of $200\sqrt{2}$ dollars for them.)

a. [4 points] For what quantities of fruit punch sold would Reggie's marginal revenue equal his marginal cost?

Answer: ______________________

b. [4 points] Assuming that Reggie can produce at most 200 gallons of fruit punch, how much fruit punch should he produce in order to maximize his profit, and what would that maximum profit be? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Answer: gallons of fruit punch: __________ and max profit: __________

6. [11 points] Ben has recently acquired a cabbage press and is opening a business selling cabbage juice. Let $R(x)$ and $C(x)$ be the revenue and cost, in dollars, of selling and producing x cups of cabbage juice. Ben only has resources to produce up to a hundred cups. After some research, Ben determines that

$$R(x) = 6x - \frac{1}{40}x^2 \qquad \text{for} \qquad 0 \leq x \leq 100$$

and

$$C(x) = \begin{cases} 60 + 2x & 0 \leq x \leq 20 \\ 70 + 1.5x & 20 < x \leq 100. \end{cases}$$

a. [3 points] What is the smallest quantity of juice Ben will need to sell in order for his profit to not be negative? Round your answer to the nearest hundredth of a cup. Show your work.

Answer: ____________________

For the following parts, determine how many cups of cabbage juice Ben needs to sell in order to maximize the given quantity. If there is no such value, write None. Use calculus to find and justify your answers.

b. [3 points] Ben's revenue.

Answer: ____________________

c. [5 points] Ben's profit.

Answer: ____________________

7. [10 points] Zerina owns a small business selling custom screen-printed and embroidered apparel.

a. Zerina receives orders for embroidered polo shirts, which she sells for \$11 each. The cost, in dollars, for her to complete an order of q embroidered polo shirts is

$$C(q) = \begin{cases} 6q - \frac{1}{8}q^2 + \frac{56}{9} & 0 \le q \le 16 \\ \frac{2}{9}q^{3/2} + 10q - 104 & q > 16. \end{cases}$$

Note that $C(q)$ is continuous for all $q \ge 0$.

i. [1 point] What is the fixed cost, in dollars, of an order of embroidered polo shirts?

Answer: ____________

ii. [5 points] Find the quantity q of embroidered polo shirts in an order that would result in the most profit for Zerina. Assume that, because of storage constraints, Zerina cannot accept an order for more than 80 embroidered polo shirts. Use calculus to find and justify your answer, and make sure you provide enough evidence to fully justify your answer.

Answer: $q =$ ____________

b. [3 points] Zerina also receives orders for screen-printed t-shirts. When a customer places such an order, they pay a \$6 setup fee, plus \$9 per t-shirt for the first 20 t-shirts ordered. Any additional t-shirts ordered only cost \$7 per t-shirt. Let $P(s)$ be the total price, in dollars, a customer pays for an order of s screen-printed t-shirts. Find a formula for $P(s)$.

Answer: $P(s) = \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_ & \text{if } 0 \le s \le 20 \\ \_\_\_\_\_\_\_\_\_\_\_\_ & \text{if } s > 20 \end{cases}$

10. [10 points] Yukiko has a small orchard where she grows Michigan apples. After careful study last season, Yukiko found that the total cost, in dollars, of producing a bushels of apples can be modeled by

$$C(a) = -25500 + 26000e^{0.002a}$$

for $0 \leq a \leq 320$.

Qabil has promised to buy up to 100 bushels of apples for his famous apple ice cream. If Yukiko has any remaining apples, she has an agreement to sell them to Xanthippe's cider mill at a reduced price. Let $R(a)$ be the revenue generated from selling a bushels of apples. Then

$$R(a) = \begin{cases} 70a & \text{if } 0 \leq a \leq 100 \\ 2000 + 50a & \text{if } 100 < a \leq 320. \end{cases}$$

a. [1 point] How much will Xanthippe's cider mill pay per bushel?

Answer: ____________________

b. [1 point] What is Yukiko's fixed cost?

Answer: ____________________

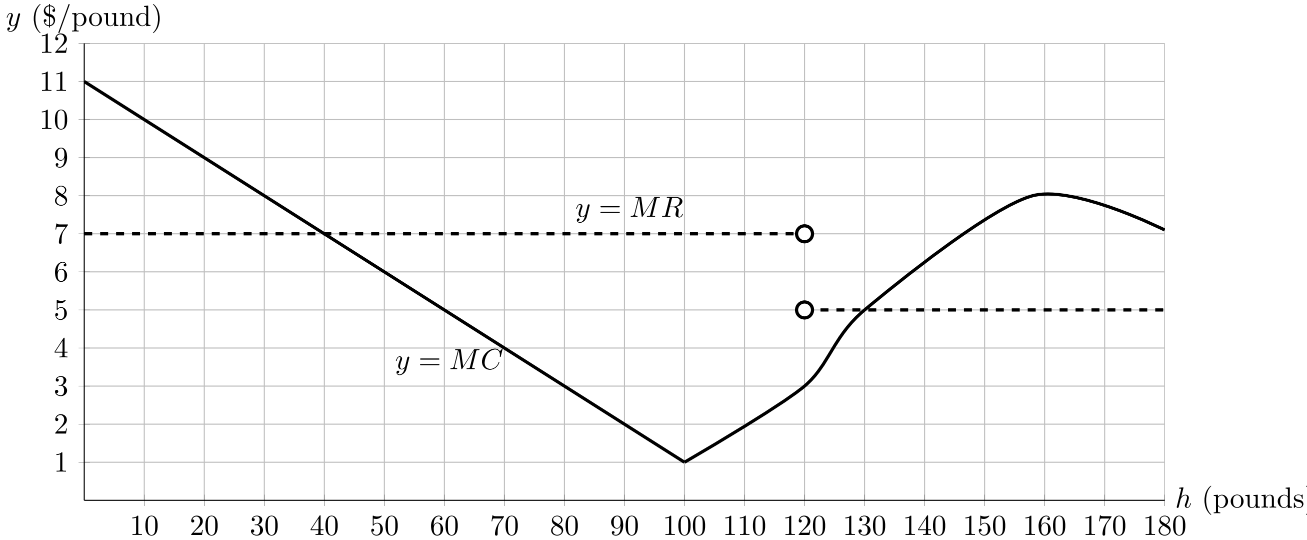
c. [4 points] For what quantities of bushels of apples sold would Yukiko's marginal revenue equal her marginal cost? Write NONE if appropriate.

Answer: ____________________

d. [4 points] Assuming Yukiko can produce up to 320 bushels of apples, how many bushels should she produce in order to maximize her profit, and what would that maximum profit be? You must use calculus to find and justify your answer. Make sure to provide enough evidence to justify your answer fully.

Answer: bushels of apples: ____________ and max profit: ____________

10. [10 points] The Happy Hives Bee Farm sells honey. The graph below shows marginal revenue MR (dashed) and marginal cost MC (solid), in dollars per pound, where h is the number of pounds of honey.



a. [7 points] Use the graph to estimate the answers to the following questions. You do not need to show work. If an answer can't be found with the information given, write "NEI".

i) For what value(s) of h in the interval $[0, 180]$ is the cost function C minimized?

Answer: $h =$ ____________________

ii) For what value(s) of h in the interval $[0, 180]$ is MC minimized?

Answer: $h =$ ____________________

iii) For what value(s) of h in the interval $[0, 180]$ is profit maximized?

Answer: h ____________________

iv) What are the fixed costs of the farm?

Answer: ____________________

v) For what values of h in the interval $[0, 180]$ is the profit function concave up?

Answer: ____________________