**9**. [12 points] Suppose $w(x)$ is an everywhere differentiable function which satisfies the following conditions:

- $w'(0) = 0$.
- $w'(x) > 0$ for $x > 0$.
- $w'(x) < 0$ for $x < 0$.

Let $f(t) = t^2 + bt + c$ where $b$ and $c$ are positive constants with $b^2 > 4c$. Define $L(t) = w(f(t))$.

**a**. [2 points] Compute $L'(t)$. Your answer may involve $w$ and/or $w'$ and constants $b$ and $c$.

Solution: $L'(t) = w'(t^2 + bt + c) \cdot (2t + b)$.

**b**. [4 points] Using your answer from (a), find the critical points of $L(t)$ in terms of the constants $b$ and $c$.

Solution: $L(t)$ has critical points when $L'(t) = 0$. This happens only if $w'(t^2+bt+c) = 0$ or if $(2t + b) = 0$.
$w'(t^2 + bt + c) = 0$ means $t^2 + bt + c = 0$ by the first property of $w'$ above. Solving using the quadratic formula, we have

$$t = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

as critical points of $L(t)$. Both of these roots exist and are distinct since $b^2 > 4c$.
If $2t + b = 0$, we have $t = -\frac{b}{2}$ as a critical point. Altogether our critical points are $t = -\frac{b}{2}, -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}, -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

**c**. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

Solution: For simplicity, let's set $p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ and $m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

We know that $f(t)$ is an upward opening parabola with roots at $p$ and $m$. We also know $p > m$, so this means $f(t) > 0$ for $t < m$ and $t > p$. This also means $f(t) < 0$ for $m < t < p$. Thus by properties two and three of $w'$ above we know $w'(f(t)) > 0$ for $t < m$ and $t > p$, and $w'(f(t)) < 0$ for $m < t < p$.

The expression $2t + b$ is positive for $t > -\frac{b}{2}$ and negative for $t < -\frac{b}{2}$.

Putting all of this information together gives us

$$L'(t) > 0$$

on the intervals $(m, -\frac{b}{2})$ and $(p, +\infty)$, and

$$L'(t) < 0$$

on the intervals $(-\infty, m)$ and $(-\frac{b}{2}, p)$. Thus, by the first derivative test, the critical points $t = m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$ and $t = p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ are local minima, and $t = -\frac{b}{2}$ is a local maximum.

**5.** [10 points] As a software engineer, Tendai spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Tendai writes in a day if he works $t$ hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

**a.** [8 points] Find the values of $t$ that minimize and maximize $w(t)$ on the interval $[1, 18]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

*Solution:* Note that $w$ is continuous at $t = 3$, since $\lim_{t\to 3^-} w(t) = \lim_{t\to 3^+} w(t) = 66$, so we may use the Extreme Value Theorem.
We find

$$w'(t) = \begin{cases} -4t + 28 & \text{if } 1 < t < 3 \\ -t + 9 & \text{if } 3 < t < 18. \end{cases}$$

The first expression is 0 when $t = 7$, but since this isn't in the domain of that piece, it is not a critical point. The second expression is 0 when $t = 9$.
Since both of these are polynomials, we don't have to worry about the derivative not existing on these open intervals. However, since $-4 \cdot 3 + 28 = 16$ and $-3 + 9 = 6$ are not equal, $w'$ is not defined at 3, so $t = 3$ is also a critical point.
Computing $w(t)$ at each critical point and the endpoints gives:

| $t$ | 1 | 3 | 9 | 18 |
|---|---|---|---|---|
| $w(t)$ | 26 | 66 | 84 | 43.5 |

By the Extreme Value Theorem, we therefore find that $w(t)$ attains its maximum value at $t = 9$ and its minimum at $t = 1$.

**Answer:** global max(es) at $t =$ 9

**Answer:** global min(s) at $t =$ 1

**b.** [2 points] What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

*Solution:* From part **a.** we see that the maximum value of $w$ is $w(9) = 84$. So according to this model, the largest number of lines of codes that Tendai can expect to write in a day is 84.

**Answer:** 84

**10**. [8 points] Consider the family of functions $g(x) = e^x - kx$, where $k$ is a positive constant.

**a**. [2 points] Show that the point $\big(\ln(k)\ ,\ k - k\ln(k)\big)$ is the only critical point of $g(x)$ for all positive $k$. Show all your work to receive full credit.

*Solution:* $g'(x) = e^x - k$, then $g'(x) = 0$ if $e^x = k$ or $x = \ln(k)$. There are no other critical points since $g'(x)$ is defined for all values of $x$.
The $y$-coordinate of the critical points is given by $g(\ln(k)) = e^{\ln(k)} - k\ln(k) = k - k\ln(k)$.

**b**. [2 points] Show that $g(x)$ has a global minimum on $(-\infty, \infty)$ at $x = \ln(k)$. Use calculus to justify your answer.

*Solution:* Since $e^x - kx \to \infty$ as $x \to \infty$ and $e^x - kx \to \infty$ as $x \to -\infty$ and $x = \ln(k)$ is the only critical point, then it is a global minimum.

**c**. [4 points] Find all values of $0.5 \leq k \leq 2$ that maximize the $y$-value of the global minimum of $g(x)$ on $(-\infty, \infty)$. Use calculus to justify your answer. Write NONE if no such value exists.

*Solution:* Let $h(k) = k - k\ln(k)$ defined on $0.5 \leq k \leq 2$. To find critical points, we find where $h'(k) = -\ln(k) = 0$. This occurs at $k = 1$. Looking at the table of values

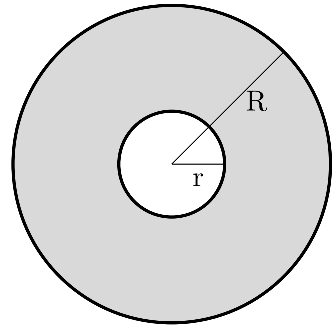
| $k$ | $0.5$ | $1$ | $2$ |
|---|---|---|---|
| $h(k)$ | $0.5(1 - \ln(0.5)) \approx 0.84$ | $1$ | $2(1 - \ln(2)) \approx 0.61$ |

Then the value of $k$ that maximizes the value of the global minimum of $g(x)$ is $k = 1$.

Answer: $k = 1$.

**8**. [8 points]

Kristen is machining a metal washer to fix her broken down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Kristen's washer is depicted by the shaded region in the figure to the right. The washer has an inner radius of $r$ centimeters and an outer radius of $R$ centimeters. The area of the washer must be exactly 5 square centimeters, and $r$ must be at least 1 centimeter.




**a**. [3 points] Find a formula for $r$ in terms of $R$.

*Solution:* The area of the washer is the difference between the outer circle's area and inner circle's area. So, since this must be 5 square centimeters we have $\pi R^2 - \pi r^2 = 5$, so $r^2 = \dfrac{\pi R^2 - 5}{\pi}$, and $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$.

**Answer:** $r =$ $\sqrt{\dfrac{\pi R^2 - 5}{\pi}}$

**b**. [2 points] The structural integrity of the washer depends on both its inner radius and its outer radius. Specifically, the structural integrity is given by the equation

$$S = 32R\left(\ln(rR + 1) + 7\right).$$

Express $S$ as a function of $R$. *Your answer should not include* $r$.

*Solution:* We substitute our answer from part a. into the formula for $S$.

**Answer:** $S(R) =$ $32R(\ln\left(R\sqrt{\dfrac{\pi R^2 - 5}{\pi}} + 1\right) + 7)$

**c**. [3 points] What is the domain of $S(R)$ in the context of this problem? You may give your answer as an interval or using inequalities.

*Solution:* We are told that $r$ must be at least 1. When $r = 1$, we have

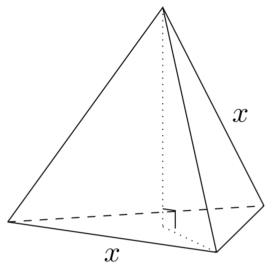
$$\pi R^2 - \pi r^2 = 5$$
$$\pi R^2 - \pi = 5$$
$$R^2 = \frac{5 + \pi}{\pi}$$
$$R = \sqrt{\frac{5 + \pi}{\pi}}.$$

This is the smallest possible value of $R$, because if we make $r$ larger, $R$ must also be made larger so that the area of the washer can remain 5 square centimeters. There is no upper bound on how large $R$ can be.

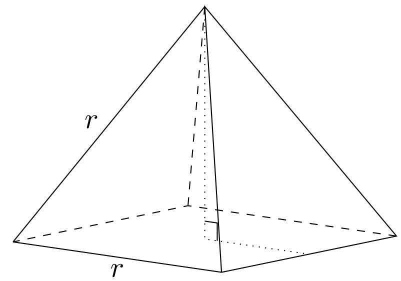
**Answer:** $[\sqrt{\dfrac{5 + \pi}{\pi}}, \infty)$

**5.** [7 points] An alien is building the wire frames of two pyramids. One has a base that is an equilateral triangle with side length $x$ meters, and the other has a base that is a square with side length $r$ meters. These shapes are shown below. For both, all triangular faces are equilateral.




Triangular Pyramid




Square Pyramid

The alien has 2 meters of wire available to build the frames, and **will use all of it**.

**a.** [2 points] Find a formula for $r$ in terms of $x$.

*Solution:* There are 6 sides of length $x$ meters, and 8 sides of length $r$ meters. In total, these lengths must add up to 2 meters, so $6x + 8r = 2$. We can then solve for $r$ in terms of $x$ and find that $r = \frac{2-6x}{8}$.

**Answer:** $r = \dfrac{2-6x}{8}$

**b.** [3 points] Find a formula for $A(x)$, the combined surface area of the two pyramids (i.e. the total area of all sides and bases of both shapes). Your formula should be in terms of $x$ only.

Recall that the area of an equilateral triangle with side length $L$ is $\dfrac{\sqrt{3}}{4}L^2$.

*Solution:* On the triangular pyramid, there are 4 equilateral triangles each with side length $x$. On the square pyramid, there are 4 equilateral triangles each with side length $r$, plus one square of side length $r$ on the base. Adding up the areas of these shapes, we find

$$\text{Total Surface Area} = 4\left(\frac{\sqrt{3}}{4}x^2\right) + 4\left(\frac{\sqrt{3}}{4}r^2\right) + r^2 = \sqrt{3}x^2 + \sqrt{3}r^2 + r^2 = \sqrt{3}x^2 + (\sqrt{3}+1)r^2.$$

We substitute $r = \dfrac{2-6x}{8}$ and find $A(x) = \sqrt{3}x^2 + \left(\sqrt{3}+1\right)\left(\frac{2-6x}{8}\right)^2$

**Answer:** $A(x) = \sqrt{3}x^2 + \left(\sqrt{3}+1\right)\left(\dfrac{2-6x}{8}\right)^2$

**c.** [2 points] The alien wants to actually build one of each type of pyramid. In the context of the problem, what is the domain of the function $A(x)$ from part **b.**? You may give your answer as an interval or using inequalities.

*Solution:* We must have $x > 0$ in order to get a triangular pyramid. We also need $r > 0$ to get a square pyramid, which in terms of $x$ means

$$\frac{2-6x}{8} > 0 \quad \text{which simplifies to} \quad 2 > 6x, \quad \text{so} \quad \frac{1}{3} > x$$

**Answer:** $\left(0, \dfrac{1}{3}\right)$