Optimizations

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1 Global Maxima and Minima

Suppose that p is a point in the domain of f:

- f has a *global minimum* at p if $f(p)$ is less than or equal to all values to all values of f.
- f has a *global maximum* at p if $f(p)$ is greater than or equal to all values to all values of f.

Global maxima and minima are sometimes called *extrema* or *optimal values.*

1.1 Existence of global extrema

Theorem 1.1 (The Extreme Value Theorem)**.** *If f is continuous on the closed interval $a \leqslant x \leqslant b$, then f has a global maximum and a global minimum on that interval.*

1.2 How to find global maxima and minima?

Global Maxima and Minima on a Closed Interval: Test the Candidates

For a continuous function f on a closed interval $a \leqslant x \leqslant b$:

- Find the critical points of f in the interval
- Evaluate the function at the critical points and at the endpoints, a and b. The largest value of the function is the global maximum; the smallest value is the global minimum.

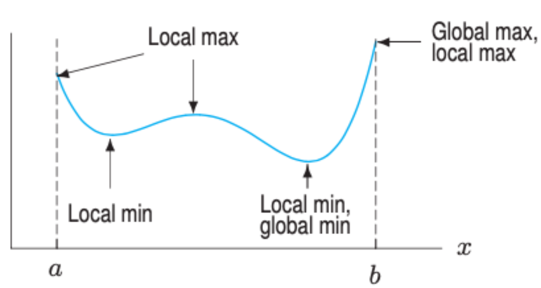


Figure 4.18: Global maximum and minimum on a closed interval $a \leq x \leq b$

Global Maxima and Minima on an Open Interval or on All Real Numbers

For a continuous function f on an open interval $a < x < b$ where a may be $-\infty$ and b may be ∞.

- Find the value of f at all the critical points in the interval.
- Look at values of f when x approaches the endpoints of the interval, or approaches $\pm\infty$, as appropriate.

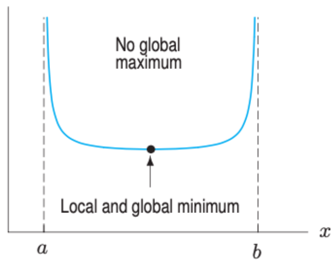


Figure 4.19: Global minimum on $a < x < b$

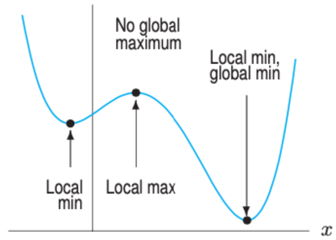


Figure 4.20: Global minimum when the domain is all real numbers

Example 1.2. Find the global maxima and minima of $f(x) = x^3 - 9x^2 - 48x + 52$ on the following intervals:

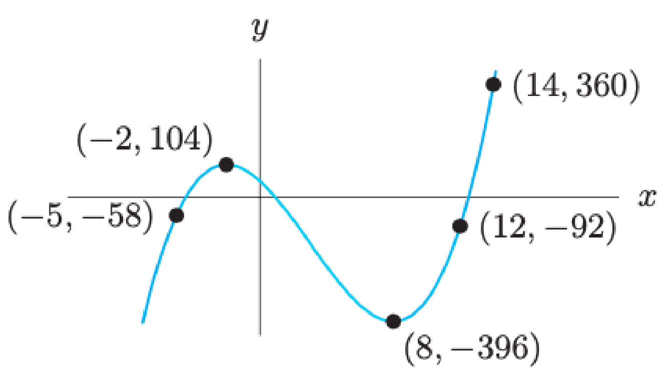
(1) $-5 \leqslant x \leqslant 12$. (2) $-5 \leqslant x \leqslant 14$. (3) $-5 \leqslant x < \infty$.

We first calculate the first derivative $f'(x) = 3x^2 - 18x - 48$. Solve for $f'(x) = 0$, we get $x = -2, 8$. These are all the critical points. We have $f(-2) = 104, f(8) = -396$.

(1) We compute $f(-5) = -58$ and $f(12) = -92$. So f has a global maximum 104 at $x = -2$ and a global minimum -396 at $x = 8$ on $[-5, 12]$.

(2) We compute $f(14) = 360$. So f has a global maximum 360 at $x = 14$ and a global minimum -396 at $x = 8$ on $[-5, 14]$.

(3) We compute $\lim_{x\to\infty} f(x) = \infty$. So f has no global maximum, a global minimum -396 at $x = 8$ on $[-5, \infty)$.



1. Find the global maximum and minimum for the function on the closed interval.
   (a) $f(x) = x^3 - 3x^2 + 20, \quad -1 \leqslant x \leqslant 3.$
   (b) $f(x) = 3x^{1/3} - x, \quad -1 \leqslant x \leqslant 8.$
   (c) $f(x) = x^2 - 2|x|, \quad -3 \leqslant x \leqslant 4.$

2. Let $f(x) = \sin^2 x - \cos x$ be defined on $0 \leqslant x \leqslant \pi$. Find the value(s) of x for which:
   (a) $f(x)$ has a local maximum or local minimum. Indicate which ones are maxima and which are minima.
   (b) $f(x)$ has a global maximum or global minimum.

Example 1.3. When an arrow is shot into the air, itr range, R, is defined as the horizontal distance from the archer to the point where the arrow hits the ground. If the ground is horizontal and we neglect air resistance, it can be shown that

$$R = \frac{v_0^2 \sin(2\theta)}{g},$$

where v_0 is the initial velocity of the arrow, g is the (constant) acceleration due to gravity, and θ is the angle above horizontal, so $0 \leqslant \theta \leqslant \pi/2$. What angle θ maximizes R?

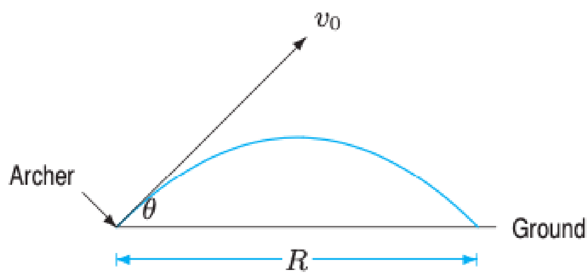


Figure 4.24: Arrow's path

Taking derivative of R with respect to θ, we get $R' = \frac{2v_0^2}{g}\cos(2\theta)$. Setting this to zero gives $\theta = \frac{\pi}{4}$. So we have to calculate

$$R(0) = 0$$
$$R(\frac{\pi}{4}) = \frac{v_0^2}{g}$$
$$R(\frac{\pi}{2}) = 0$$

So $\theta = \frac{\pi}{4}$ maximizes R, and the maximum value is $\frac{v_0^2}{g}$.

1. For positive constants A and B, the force between two atoms in a molecule is given by

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3},$$

   What value of r minimizes the force between the atoms?

Example 1.4. For a positive constant b, the surge function $f(t) = t\mathrm{e}^{-bt}$ gives the quantity of a drug in the body for time $t \geqslant 0$. Find the global maximum and minimum of $f(t)$ for $t \geqslant 0$.

(1) Find the global maximum and minimum of $f(t)$ for $t \geqslant 0$.

(2) Find the value of b making $t = 10$ the global maximum.

The first derivative is $f'(t) = (1 - bt)\mathrm{e}^{-bt}$. So $f'(t) = 0 \Rightarrow t = \frac{1}{b}$. For (1), we compute $f(0) = 0, f(\frac{1}{b}) = \frac{1}{b\mathrm{e}}$. Also we have to look at the end behaviour of $f(t)$ as $t \to \infty$, which is $\lim_{t\to\infty} f(t) = 0$. So the global minimum is $f(t) = 0$ at $t = 0$ and the global maximum is $\frac{1}{b\mathrm{e}}$ at $t = \frac{1}{b}$.

For (2), we set $\frac{1}{b} = 10 \Leftrightarrow b = 1/10$.