3. [11 points] For positive constants a and b, the potential energy of a particle is given by

$$U(x) = a\left(\frac{5b^2}{x^2} - \frac{3b}{x}\right).$$

Assume that the domain of $U(x)$ is the interval $(0, \infty)$.

a. [2 points] Find the asymptotes of $U(x)$. If there are none of a particular type, write NONE.

Solution: We can get a common denominator and write

$$U(x) = a\frac{5b^2 - 3bx}{x^2}.$$

We see that there is a vertical asymptote at $x = 0$, where the denominator is zero, and a horizontal asymptote at $U = 0$, since the degree of the denominator is greater than the degree of the numerator.

Answer: Vertical asymptote(s): $x = 0$ Horizontal asymptote(s): $U = 0$

b. [6 points] Find the x-coordinates of all local maxima and minima of $U(x)$ in the domain $(0, \infty)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Solution: First we find critical points by looking at where $U'(x)$ is undefined or zero. We have

$$U'(x) = a\left(-\frac{10b^2}{x^3} + \frac{3b}{x^2}\right) = a\frac{3bx - 10b^2}{x^3}.$$

There are no points in the domain of $U(x)$ where $U'(x)$ is undefined, but $U'(x)$ has a zero where $3bx - 10b^2 = 0$, or $x = \frac{10b}{3}$.
To classify this critical point, we can use the First or Second Derivative Test. We will use the Second Derivative Test here, so we compute

$$U''(x) = a\left(\frac{30b^2}{x^4} - \frac{6b}{x^3}\right) = \frac{6ab}{x^4}(5b - x).$$

Since a, b, and x^4 are always positive and $5b - x$ is positive at $x = \frac{10b}{3}$, we see that $U''\left(\frac{10b}{3}\right) > 0$, and hence $x = \frac{10b}{3}$ is a local minimum.
There are no other critical poitns to consider, so there are no local maxima.

Answer: Local max(es) at $x =$ NONE Local min(s) at $x =$ $\frac{10b}{3}$

c. [3 points] Suppose $U(x)$ has an inflection point at $(6, -14)$. Find the values of a and b. *Show your work, but you do not need to verify that this point is an inflection point.*

Solution: We already found $U''(x) = \frac{6ab}{x^4}(5b - x)$, so we see that the only potential inflection point occurs at $x = 5b$, the only place in the domain of $U(x)$ where $U''(x)$ is zero or undefined. Hence $5b = 6$ or $b = 1.2$.
Plugging in $x = 6$, $b = 1.2$, and $U = -14$ into the original equation for $U(x)$ yields

$$-14 = a\left(\frac{1}{5} - \frac{3}{5}\right) = -\frac{2a}{5}$$

and hence $a = 35$.

Answer: $a =$ 35 and $b =$ 1.2

3. [13 points] Let f be a function such that $f''(x)$ is defined for all real numbers. A table of some values of f' is given below.

x	2	3	4	6	9	11
$f'(x)$	4	1	0	2	0	-4

Assume that f' is continuous and either always decreasing or always increasing between consecutive values of x shown in the table.

a. [2 points] Using the table above, estimate $f''(11)$. *Show your work.*

Solution: Since f'' is the derivative of f', $f''(11) \approx \dfrac{f'(11) - f'(9)}{11 - 9} = \dfrac{-4 - 0}{11 - 9} = -2$.

Answer: $f''(11) \approx$ -2

For parts (b) through (e) below, find the indicated values.
Write NONE if there are no such values of x.
Write NOT ENOUGH INFO if there is not sufficient information to determine a value.
You do not need to explain your reasoning.

b. [3 points] Find the x-coordinates of all critical points of $f(x)$ on the interval $2 < x < 11$.

Answer: critical point(s) at $x =$ 4, 9

c. [3 points] Find the x-coordinates of all local minima of $f(x)$ on the interval $2 < x < 11$.

Answer: local min(s) at $x =$ NONE

d. [3 points] Find the x-coordinates of all inflection points of $f(x)$ on the interval $2 < x < 11$.

Answer: inflection point(s) at $x =$ 4, 6

e. [2 points] Find all values of x at which $f(x)$ attains its global maximum on the interval $2 \leq x \leq 11$.

Answer: global max(es) at $x =$ 9

7. [9 points] Consider the family of functions given by $f(x) = e^{x^2+Ax+B}$ for constants A and B.

a. [6 points] Find and classify all local extrema of $f(x) = e^{x^2+Ax+B}$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate. Your answers may depend on A and/or B.

Solution: First, we find the critical points of $f(x)$. Notice that $f(x)$ is differentiable, so $f(x)$ only has critical points where $f'(x) = 0$. Since

$$f'(x) = (2x + A)e^{x^2+Ax+B},$$

the only critical point of $f(x)$ occurs where $2x+A = 0$, i.e., at $x = -\frac{A}{2}$. We test whether this critical point is a local maximum, a local minimum, or neither.
Applying the First Derivative Test:

- For $x < -\frac{A}{2}$: $2x + A < 0$ and $e^{x^2+Ax+B} > 0$, so $f'(x) < 0$.
- For $x > -\frac{A}{2}$: $2x + A > 0$ and $e^{x^2+Ax+B} > 0$, so $f'(x) > 0$.

Hence, $f'(x)$ changes from negative to positive at $x = -\frac{A}{2}$, so $f(x)$ has a local minimum at $x = -\frac{A}{2}$ (and no local maxima).
(Note that we could instead apply the Second Derivative Test: $f''(x) = (2x + A)(2x + A)(e^{x^2+Ax+B} + 2(e^{x^2+Ax+B}) = \left((2x + A)^2 + 2\right) e^{x^2+Ax+B}$ which is always positive (since both factors are always positive). So in particular $f''(-A/2) > 0$ so $f(x)$ has a local minimum at $x = -A/2$.

Answer: Local min(s) at $x =$ $-\frac{A}{2}$

Answer: Local max(es) at $x =$ NONE

b. [3 points] Find exact values of the constants A and B so that the point $(3, 1)$ is a critical point of $f(x) = e^{x^2+Ax+B}$.

Solution: As we showed in part **a.**, $f(x)$ has its only critical point at $x = -\frac{A}{2}$, so now we must have that $-\frac{A}{2} = 3$ so $A = -6$.
To find B, we use the fact that $(3, 1)$ is a point on the graph of $y = f(x)$ (so $f(3) = 1$).

$$\begin{aligned} 1 &= f(3) = e^{3^2+(-6)(3)+B} = e^{B-9} \\ \ln(1) &= \ln\left(e^{B-8}\right) \\ 0 &= B - 9 \\ B &= 9 \end{aligned}$$

Hence, $A = -6$ and $B = 9$.

Answer: $A =$ -6 and $B =$ 9