# Local Extrema and Second Derivative Test

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March 10, 2020

## 1 Local Extrema and Critical Points

Recall that if $p$ is a point in the domain of $f$:

- $f$ has a *local minimum* at $p$ if $f(p)$ is less than or equal to the values of $f$ for points near $p$.
- $f$ has a *local maximum* at $p$ if $f(p)$ is greater than or equal to the values of $f$ for points near $p$.
- $f$ has a *critical point* at $p$ if $f'(p) = 0$ or $f'(p)$ is undefined.

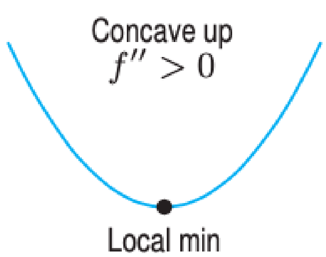
For finding local extremas, we can use the first derivative test (see notes from last class).

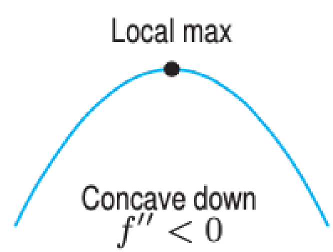
## 2 Second Derivative Test

The Second-Derivative Test for Local Maxima and Minima:

Suppose $p$ is a critical point of a continuous function $f$.

- If $f'(p) = 0$ and $f''(p) > 0$ then $f$ has a local minimum at $p$.
- If $f'(p) = 0$ and $f''(p) < 0$ then $f$ has a local maximum at $p$.
- If $f'(p) = 0$ and $f''(p) = 0$ then the test tells us nothing.




**Example 2.1.** Suppose that $f(x) = x^3 - 9x^2 - 48x + 52$. How do we find all local extrema?

First we need to find all critical points. For that purpose, we take the derivative of the function, which is $3x^2 - 18x - 48$. Then we find all points such that the derivative is zero at that point. This is a quadratic function and hence we can use quadratic formula to find that two roots are $x = -2, 8$.

Now we use the second derivative test. We compute $f''(x) = 6x - 18$. So $f''(-2) = -30 < 0$ and $f''(8) = 30 > 0$. So $x = -2$ is a local maximum, and $x = 8$ is a local minimum.

1. Using the second-derivative test to determine local maxima and minima. Check your answer by graphing.
   (a) $f(x) = 3x^4 - 4x^3 + 6$.
   (b) $f(x) = e^{-2x^2}$.
   (c) $f(x) = 2x - 5\ln(x)$.

# 3 Inflection Points

## 3.1 In terms of concavity

**Definition 3.1.** A point $p$ at which the graph of a continuous function $f$ changes concavity is called an *inflection point* of $f$.

## 3.2 In terms of the first derivative

Suppose a function f has a continuous derivative. If $f''$ changes sign at $p$, then $f$ has an inflection point at $p$, and $f'$ has a local extremum at $p$.

## 3.3 How do we detect an inflection point?

Suppose $f''$ is defined on both sides of a point $p$:

- If $f''$ is zero or undefined at $p$, then $p$ is a possible inflection point.
- To test whether $p$ is an inflection point, check whether $f''$ changes sign at $p$.

**Example 3.2.** For $x \geqslant 0$, we want to find the local maxima and minima and inflection points for $g(x) = xe^{-x}$ and sketch the graph of $g$.

Taking derivative we have $g'(x) = (1-x)e^{-x}$ and $g''(x) = (x-2)e^{-x}$. So $g'(x) = 0 \Rightarrow x = 1$ is the only ciritical point. And at $x = 1$, $g''(1) = -e^{-1} < 0$. Hence $g$ has a local maximum at $x = 1$.

Setting $g''(x) = 0$, we get $x = 2$. Draw a table as follows:

| $x$ | $x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|
| $g''(x)$ | $-$ | 0 | $+$ |

So $x = 2$ is an inflection point of $g$.

1. Find all inflection points of $y = x^4$.

**3**. [11 points] For positive constants $a$ and $b$, the potential energy of a particle is given by

$$U(x) = a\left(\frac{5b^2}{x^2} - \frac{3b}{x}\right).$$

Assume that the domain of $U(x)$ is the interval $(0, \infty)$.

**a**. [2 points] Find the asymptotes of $U(x)$. If there are none of a particular type, write NONE.

**Answer:** Vertical asymptote(s): __________ Horizontal asymptote(s): __________

**b**. [6 points] Find the $x$-coordinates of all local maxima and minima of $U(x)$ in the domain $(0, \infty)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

**Answer:** Local max(es) at $x =$ __________ Local min(s) at $x =$ __________

**c**. [3 points] Suppose $U(x)$ has an inflection point at $(6, -14)$. Find the values of $a$ and $b$. *Show your work, but you do not need to verify that this point is an inflection point.*

**Answer:** $a =$ __________ and $b =$ __________

**3**. [13 points] Let $f$ be a function such that $f''(x)$ is defined for all real numbers. A table of some values of $f'$ is given below.

| $x$ | 2 | 3 | 4 | 6 | 9 | 11 |
|---|---|---|---|---|---|---|
| $f'(x)$ | 4 | 1 | 0 | 2 | 0 | -4 |

**Assume that $f'$ is continuous and either always decreasing or always increasing between consecutive values of $x$ shown in the table.**

**a**. [2 points] Using the table above, estimate $f''(11)$. *Show your work.*

**Answer:** $f''(11) \approx$ ____________________

For parts (b) through (e) below, find the indicated values.
Write NONE if there are no such values of $x$.
Write NOT ENOUGH INFO if there is not sufficient information to determine a value.
You do not need to explain your reasoning.

**b**. [3 points] Find the $x$-coordinates of all critical points of $f(x)$ on the interval $2 < x < 11$.

**Answer:** critical point(s) at $x =$ ____________________

**c**. [3 points] Find the $x$-coordinates of all local minima of $f(x)$ on the interval $2 < x < 11$.

**Answer:** local min(s) at $x =$ ____________________

**d**. [3 points] Find the $x$-coordinates of all inflection points of $f(x)$ on the interval $2 < x < 11$.

**Answer:** inflection point(s) at $x =$ ____________________

**e**. [2 points] Find all values of $x$ at which $f(x)$ attains its global maximum on the interval $2 \leq x \leq 11$.

**Answer:** global max(es) at $x =$ ____________________

**7**. [9 points] Consider the family of functions given by $f(x) = e^{x^2+Ax+B}$ for constants $A$ and $B$.

**a**. [6 points] Find and classify all local extrema of $f(x) = e^{x^2+Ax+B}$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate. Your answers may depend on $A$ and/or $B$.

**Answer:** Local min(s) at $x =$ ________________________

**Answer:** Local max(es) at $x =$ ________________________

**b**. [3 points] Find exact values of the constants $A$ and $B$ so that the point $(3, 1)$ is a critical point of $f(x) = e^{x^2+Ax+B}$.

**Answer:** $A =$ ____________ and $B =$ ____________