Local Extrema, Critical Points and Mean Value Theorem

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1 First derivative

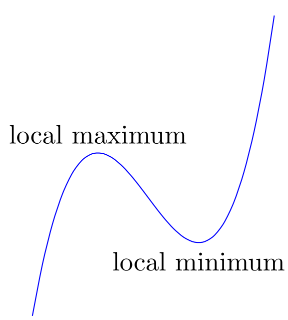
We've learned a lot on taking derivatives. Now we are in a position to learn how to make use of these derivatives. Let us start with first derivative.

1.1 Local extrema

Suppose p is a point in the domain of f:

- f has a *local minimum* at p if $f(p)$ is less than or equal to the values of f for points near p.
- f has a *local maximum* at p if $f(p)$ is greater than or equal to the values of f for points near p.

We use the adjective "local" because we are describing only what happens near p. Local maxima and minima are sometimes called *local extrema*.



1.2 Critical points

Definition 1.1. For any function f, a point p in the domain of f where $f'(p) = 0$ or $f'(p)$ is undefined is called a *critical point* of the function. In addition, the point $(p, f(p))$ on the graph of f is also called a *critical point*. A critical value of f is the value, $f(p)$, at a critical point, p.

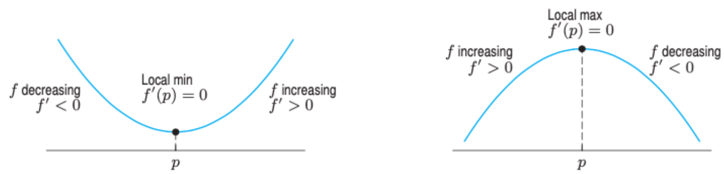
Theorem 1.2. *Suppose f is defined on an interval and has a local maximum or minimum at the point $x = a$, which is not an endpoint of the interval. If f is differentiable at $x = a$, then $f'(a) = 0$. Thus, a is a critical point.*

1.3 First derivative test

The First-Derivative Test for Local Maxima and Minima:

Suppose p is a critical point of a continuous function f . Moving from left to right:

- If f' changes from negative to positive at p, then f has a local minimum at p.
- If f' changes from positive to negative at p, then f has a local maximum at p.



Example 1.3. Suppose that $f(x) = x^3 - 9x^2 - 48x + 52$. How do we find all local extrema?

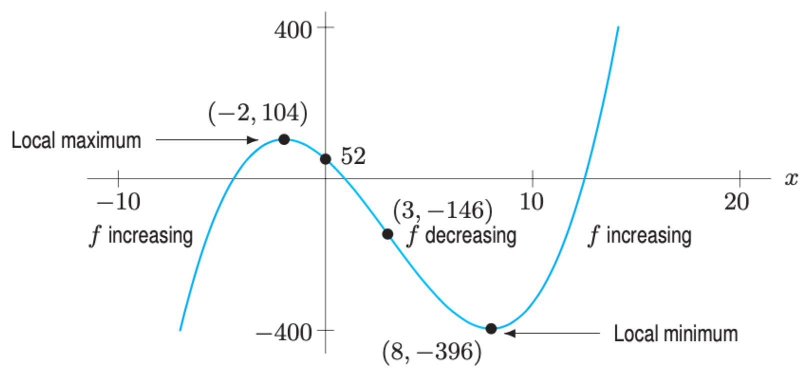
First we need to find all critical points. For that purpose, we take the derivative of the function, which is $3x^2 - 18x - 48$. Then we find all points such that the derivative is zero at that point. This is a quadratic function and hence we can use quadratic formula to find that two roots are $x = -2, 8$.

Now we use the first derivative test. So we draw a table as below

x	$x < -2$	$-2 < x < 8$	$x > 8$
$f'(x)$	+	−	+

By first derivative test, we see that $x = -2$ is a local maximum, and $x = 8$ is a local minimum.

Let's check the claim by looking at the graph



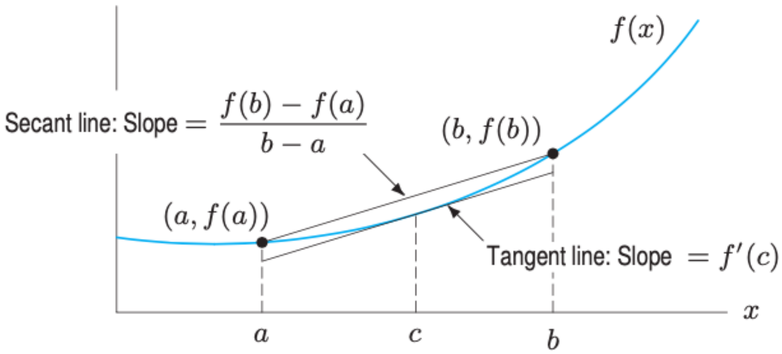
1. find all critical points and then use the first derivative test to determine local maxima and minima. Draw a graph to verify your answer.
   (a) $f(x) = 3x^4 - 4x^3 + 6$.
   (b) $f(x) = \dfrac{x}{x^2 + 1}$

2 Mean value theorem

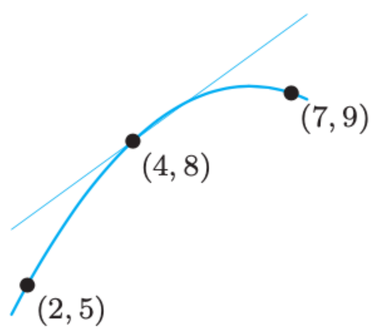
Theorem 2.1 (The Mean Value Theorem). *If f is continuous on $a \leqslant x \leqslant b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

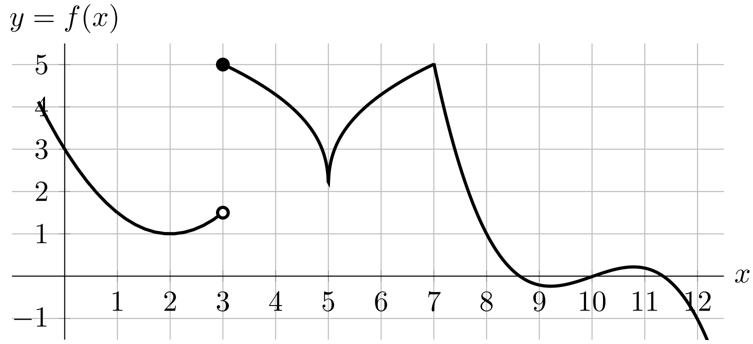
In other words, $f(b) - f(a) = f'(c)(b - a)$.



1. Applying the Mean Value Theorem with $a = 2, b = 7$ to the function below leads to $c = 4$. What is the equation of the tangent line at 4?



6. [4 points] The graph of the function $f(x)$ is shown below. Note that $f(x)$ has a vertical tangent line at $x = 5$.

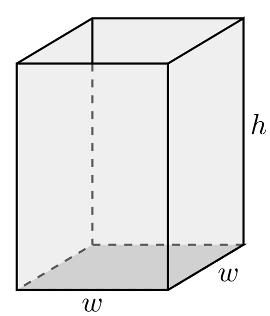


a. [2 points] On which of the following intervals does the function $f(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle the correct answer(s).

[0,2] [1,3] [2,4] [3,5] NONE OF THESE

b. [2 points] On the interval $[8, 12]$ the hypotheses of the Mean Value Theorem are true for the function $f(x)$. What does the conclusion of this theorem say in this interval?
Answer:

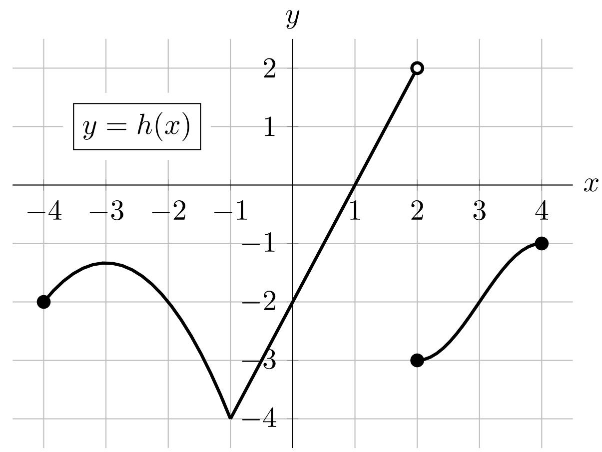
7. [5 points]
Yi is constructing a cardboard box. The base of the box will be a square of width w inches. The height of the box will be h inches. Yi will use gray cardboard for the sides of the box and brown cardboard for the bottom (the box does not have a top). Gray cardboard costs \$0.05 per square inch, while brown cardboard costs \$0.03 per square inch. Yi wants to spend \$20 on the cardboard for his box.



Write a formula for h in terms of w.

Answer: $h =$ ______________________

7. [11 points]
Shown to the right is the graph of a function $h(x)$.



For parts **a.–c.**, circle **all** correct choices.

a. [2 points] Which of the following are critical points of $h(x)$?

$x = -3$ $\quad$ $x = -1$ $\quad$ $x = 1$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ NONE OF THESE

b. [2 points] On which of the following interval(s) does $h(x)$ satisfy the hypotheses of the Mean Value Theorem?

$[-4, -1]$ $\quad$ $[-4, 0]$ $\quad$ $[0, 2]$ $\quad$ $[3, 4]$ $\quad$ NONE OF THESE

c. [2 points] On which of the following interval(s) does $h(x)$ satisfy the conclusion of the Mean Value Theorem?

$[-4, -1]$ $\quad$ $[-4, 0]$ $\quad$ $[0, 2]$ $\quad$ $[3, 4]$ $\quad$ NONE OF THESE

d. [5 points] Define the function $k(x)$ such that

$$k(x) = \begin{cases} h(x) & -4 \leq x < 1 \\ A^2 \sin(Ax + B) & 1 \leq x \leq 4, \end{cases}$$

where A and B are constants. Find one pair of values for A and B that make $k(x)$ differentiable at $x = 1$. *Show your work.*

Answer: $A =$ ____________ and $B =$ ____________