1. [5 points] Let $h(x)$ be a differentiable function such that $h'(x)$ is also differentiable everywhere. Suppose that $h(3) = 9$, $h'(3) = 2$, and $h''(x) > 0$ for all real numbers x.

a. [2 points] Let $L(x)$ be the local linearization of $h(x)$ at $x = 3$. Find a formula for $L(x)$.

Solution: The graph of $L(x)$ is the tangent line to the graph of $y = h(x)$ at $x = 3$. This is a line of slope 2 passing through the point $(3, 9)$. So $L(x) = 9 + 2(x - 3)$.

Answer: $L(x) =$ $9 + 2(x - 3)$

b. [3 points] Which of the following equalities could be true?
Circle all the statements that could be true or circle NONE OF THESE.
You do not need to explain your reasoning.

Solution: Since $h''(x) > 0$ for all x, the graph of $h(x)$ is concave up so lies above the graph of $L(x)$. Therefore, $h(-1) > L(-1) = 9 + 2(-4) = 1$.

$h(-1) = -1$

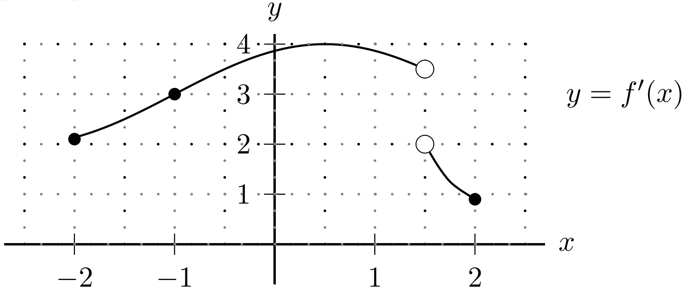
$h(-1) = 0$

$h(-1) = 1$

$\boxed{h(-1) = 2}$

NONE OF THESE

3. [8 points] Suppose $f(x)$ is a function that is continuous on the interval $[-2, 2]$. The graph of $f'(x)$ on the interval $[-2, 2]$ is given below.



a. [3 points] Let $L(x)$ be the local linearization of $f(x)$ at $x = -1$. Using the fact that $f(-1) = 4$, write a formula for $L(x)$.

Solution: $f(-1) = 4$ and $f'(-1) = 3$, so $L(x) = 4 + 3(x - (-1)) = 4 + 3(x + 1)$.

Answer: $L(x) =$ $4 + 3(x + 1)$ **or** $3x + 7$

b. [2 points] Use your formula for $L(x)$ to approximate $f(-0.5)$.

Solution: Since -0.5 is close to -1 we have

$$f(-0.5) \approx L(-0.5) = 4 + 3(-0.5 + 1) = 43(0.5) = 5.5.$$

Answer: $f(-0.5) \approx$ 5.5

c. [3 points] Is your answer from part (b) an overestimate or an underestimate of the actual value of $f(-0.5)$? Justify your answer.

Circle one: overestimate [✓ underestimate] CANNOT BE DETERMINED

Justification:

Solution: The function $f'(x)$ is increasing between -2 and 0 so $f(x)$ is concave up over this interval. Therefore the tangent line to the graph of $f(x)$ at $x = -1$ lies below the graph of $f(x)$ between $x = -2$ and $x = 0$. In particular, the local linearization $L(x)$ of $f(x)$ at $x = -1$ gives an underestimate of f on that interval.

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Solution: The closest point in the table to $v = 64$ is $v = 60$, so this is the appropriate choice for the tangent line approximation. Based on the table, the line will go through $(60, 84)$ and have slope 1.8, so it must be $L(v) = 84 + 1.8(v - 60)$. Plugging in 64 for v, we get an estimate of 91.2 seconds.

Answer: 91.2 seconds

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: Let $Q(v)$ be the quadratic approximation of $T(v)$ at $v = 200$. Then

$$Q(v) = T(200)+T'(200)(v-200)+\frac{T''(200)}{2}(v-200)^2 = 513+0.9(v-200)+\frac{-0.05}{2}(v-200)^2.$$

So the resulting approximation of $T(205)$ is given by

$$T(205) \approx Q(205) = 513+0.9(205-200)-\frac{0.05}{2}(205-200)^2 = 513+4.5-0.625 = 516.875.$$

Answer: $T(205) \approx$ 516.875

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should not include the function names T or C.

Solution: We know $L(v) = C(100)+C'(100)(v-100)$. We also know $C(100) = T(10) = 11$. So we need to find $C'(100)$.

Since $C(v) = T(\sqrt{v})$, we apply the chain rule and see that $C'(v) = \frac{1}{2\sqrt{v}}T'(\sqrt{v})$. Using the table above, we then find that $C'(100) = \frac{1}{20}T'(10) = \frac{2.4}{20} = 0.12$.

So $L(v) = 11 + 0.12(v - 100)$.

Answer: $L(v) =$ $11 + 0.12(v - 100)$

8. [14 points]

Suppose H is a differentiable function such that $H'(w)$ is also differentiable for $0 < w < 10$. Several values of $H(w)$ and of its first and second derivatives are given in the table on the right.

w	1	2	3	5	8
$H(w)$	6.3	5.4	5.2	4.8	0.7
$H'(w)$	-1.5	-0.4	-0.1	-0.6	-2.1
$H''(w)$	1.6	0.9	0	-0.8	-0.4

Assume that between each pair of consecutive values of w shown in the table, each of $H'(w)$ and $H''(w)$ is either always strictly decreasing or always strictly increasing. *Remember to show your work carefully.*

a. [3 points] Use an appropriate linear approximation to estimate $H(5.2)$.

Solution: For w near 5, local linearization gives $H(w) \approx H(5) + H'(5)(w - 5)$, so

$$H(5.2) \approx H(5) + H'(5)(5.2 - 5) = 4.8 - 0.6(0.2) = 4.8 - 0.12 = 4.68.$$

Answer: $H(5.2) \approx$ 4.68

b. [5 points] Let $J(w)$ be the local linearization of H near $w = 2$, and let $K(w)$ be the local linearization of H near $w = 3$. Which of the following statements must be true? Circle all of the statements that must be true, or circle "NONE OF THESE".

$J(2) > H(2)$

$\boxed{J(2) = H(2)}$

$J(2) < H(2)$

$J'(2) > H'(2)$

$\boxed{J'(2) = H'(2)}$

$J'(2) < H'(2)$

$J(2.5) > H(2.5)$

$J(2.5) = H(2.5)$

$\boxed{J(2.5) < H(2.5)}$

$K(2.5) > H(2.5)$

$K(2.5) = H(2.5)$

$\boxed{K(2.5) < H(2.5)}$

$\boxed{K(3.5) > H(3.5)}$

$K(3.5) = H(3.5)$

$K(3.5) < H(3.5)$

$\boxed{K'(3.5) > H'(3.5)}$

$K'(3.5) = H'(3.5)$

$K'(3.5) < H'(3.5)$

NONE OF THESE

c. [3 points] Use the quadratic approximation of $H(w)$ at $w = 1$ to estimate $H(0.9)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: Let $Q(w)$ be the quadratic approximation of $H(w)$ at $w = 1$.
Then $Q(w) = H(1) + H'(1)(w - 1) + \frac{H''(1)}{2}(w - 1)^2 = 6.3 - 1.5(w - 1) + \frac{1.6}{2}(w - 1)^2$.
So, $H(0.9) \approx Q(0.9) = 6.3 - 1.5(0.9 - 1) + \frac{1.6}{2}(0.9 - 1)^2 = 6.3 + 0.15 + 0.008 = 6.458$.

Answer: $H(0.9) \approx$ 6.458

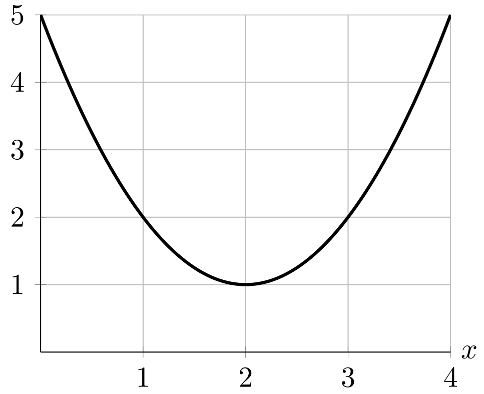
d. [3 points] Consider the function N defined by $N(w) = H(2w^2 - 10)$, and let $L(w)$ be the local linearization of $N(w)$ at $w = 3$. Find a formula for $L(w)$. Your answer should not include the function names N or H.

Solution: We know that $L(w) = N(3) + N'(3)(w - 3)$.
Note that $N(3) = H(2(3^2) - 10) = H(8) = 0.7$.
To find $N'(3)$, we apply the Chain Rule. In particular, $N'(w) = (4w) \cdot H'(2w^2 - 10)$, so $N'(3) = (4 \cdot 3) \cdot H'(2(3^2) - 10) = 12H'(8) = 12(-2.1) = -25.2$.
Therefore, $L(w) = N(3) + N'(3)(w - 3) = 0.7 - 25.2(w - 3)$.

Answer: $L(w) =$ $0.7 - 25.2(w - 3)$

9. [7 points] The graph of $h'(x)$ (the **derivative** of $h(x)$) is shown below.

$y = h'(x)$



a. [3 points] Find a formula for the tangent line approximation $L(x)$ to the function $h(x)$ near $x = 2$ if the point $(2, -3)$ lies on the graph of $y = h(x)$. Your answer should not include the letter h.

Solution: $h(2) = -3$ and $h'(2) = 1$.

Answer: $L(x) = -3 + (x - 2)$

b. [1 point] Use the tangent line approximation to the function $h(x)$ near $x = 2$ to approximate the value of $h(1.6)$.

Solution:

Answer: $h(1.6)$ is approximately $L(1.6) = -3 + (1.6 - 2) = -3.4$.

c. [3 points] Is your approximation in part **b** an overestimate or an underestimate? Circle your answer and give a justification of your answer.

Solution:

[✓ Overestimate] Underestimate Not enough information

Justification:

Since $h'(x)$ is decreasing on $[1.6, 2]$, $h(x)$ is concave down on $[1.6, 2]$. Hence the approximation is an overestimate.

10. [9 points] Consider the function h defined by $$h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80\ln\left(\frac{x}{2}\right) & \text{if } x \geq 2 \end{cases}$$ where A and B are constants.

a. [6 points] Find values of A and B so that h is differentiable.
Remember to show your work clearly.

Solution: If h is differentiable, it must be continuous, so, in particular,

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^+} h(x) \\ A(2)^4 &= B(2)^3 + 80\ln(2/2) \\ 16A &= 8B \\ 2A &= B. \end{aligned}$$

Note that $\frac{d}{dx}\left(Ax^4\right) = 4Ax^3$ and $\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right) = 3Bx^2 + 80(\frac{2}{x})(\frac{1}{2}) = 3Bx^2 + \frac{80}{x}$. and that both Ax^4 and $Bx^3 + 80\ln\left(\frac{x}{2}\right)$ are differentiable at $x = 2$.
In order for $h(x)$ to be differentiable at $x = 2$, $h'(x)$ must exist at $x = 2$. In particular,

$$\begin{aligned} \lim_{k\to 0^-} \frac{h(2+k) - h(2)}{k} &= \lim_{k\to 0^+} \frac{h(2+k) - h(2)}{k} \\ \left(\frac{d}{dx}(Ax^4)\right)\Bigg|_{x=2} &= \left(\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right)\right)\Bigg|_{x=2} \\ (4Ax^3)\Big|_{x=2} &= \left(3Bx^2 + \frac{80}{x}\right)\Bigg|_{x=2} \quad \text{(i.e. derivatives of the two pieces are equal at } x = 2\text{)} \\ 32A &= 12B + 40. \end{aligned}$$

Since $B = 2A$, we therefore find that

$$\begin{aligned} 32A &= 24A + 40 \\ 8A &= 40 \\ A &= 5 \end{aligned}$$

and hence $B = 2A = 2(5) = 10$.

Answer: $A =$ 5 and $B =$ 10

b. [3 points] Using the values of A and B you found in part **a.**, find the tangent line approximation for $h(x)$ near $x = 1$.

Solution: First, notice that

$$h(1) = 5(1)^4 = 5$$

and

$$h'(1) = 4(5)(1)^3 = 20.$$

So the tangent line approximation for $h(x)$ near $x = 1$ is $y = 5 + 20(x-1) = 20x - 15$.

Answer: The tangent line approximation is given by $y =$ $5 + 20(x-1)$ **(or** $20x - 15$**)**

11. [6 points] Let $h(x) = x^x$. For this problem, it may be helpful to know the following formulas:

$$h'(x) = \; x^x\left(\ln(x) + 1\right) \qquad \text{and} \qquad h''(x) = \; x^x\left(\frac{1}{x} + (\ln(x) + 1)^2\right).$$

a. [2 points] Write a formula for $p(x)$, the local linearization of $h(x)$ near $x = 1$.

Solution: $h(1) = 1$ and $h'(1) = 1^1(\ln(1) + 1) = 1$, so $p(x) = 1 + 1 \cdot (x - 1) = x$.

Answer: $p(x) =$ x

b. [4 points] Write a formula for $u(x)$, the quadratic approximation of $h(x)$ at $x = 1$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: $h''(1) = 1(1 + (0 + 1)^2) = 2$, so $u(x) = 1 + (x - 1) + \frac{2}{2}(x - 1)^2 = x^2 - x + 1$.

Answer: $u(x) =$ $1 + (x - 1) + (x - 1)^2 \;\; (= x^2 - x + 1)$