Linear and Quadratic Approximations

Zhan Jiang

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1 Approximations

1.1 Linear Approximation

Definition 1.1 (The Tangent Line Approximation)**.** Suppose f is differentiable at a. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a).$$

The expression $f(a) + f'(a)(x - a)$ is called the local linearization of f near $x = a$. We are thinking of a as fixed, so that $f(a)$ and $f'(a)$ are constants.

The error, $E(x)$, in the approximation is defined by

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

1. Find linear approximations of following functions near the given value.
   (a) $\sqrt{1+x}$ near $x = 0$.
   (b) e^x near $x = 0$.
   (c) $1/x$ near $x = 1$.
   (d) e^{x^2} near $x = 1$.
   (e) $\ln(1 + x)$ near $x = 0$.
   (f) $(1 + x)^n$ near $x = 0$.

1.2 Quadratic Approximation

Given a function $f(x)$, we want to find a quadratic function $Q(x) = ax^2 + bx + c$ at a certain point x_0 such that $Q(x_0) = f(x_0), Q'(x_0) = f'(x_0), Q''(x_0) = f''(x_0)$. Let us assume that $f(x_0) = A, f'(x_0) = B$ and $f''(x_0) = C$. Then what are a, b, c for $Q(x)$?

Definition 1.2 (Quadratic Approximation)**.** Suppose f is differentiable at a. Then, for values of x near a, the quadratic approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x-a) + \underline{\hspace{4cm}}.$$

1. Find quadratic approximations of following functions near the given value.
   (a) $\sqrt{1+x}$ near $x = 0$.
   (b) e^x near $x = 0$.
   (c) $1/x$ near $x = 1$.
   (d) e^{x^2} near $x = 1$.
   (e) $\ln(1+x)$ near $x = 0$.
   (f) $(1+x)^n$ near $x = 0$.

1. [5 points] Let $h(x)$ be a differentiable function such that $h'(x)$ is also differentiable everywhere. Suppose that $h(3) = 9$, $h'(3) = 2$, and $h''(x) > 0$ for all real numbers x.

a. [2 points] Let $L(x)$ be the local linearization of $h(x)$ at $x = 3$. Find a formula for $L(x)$.

Answer: $L(x) =$ ____________________________

b. [3 points] Which of the following equalities could be true?
Circle all the statements that could be true or circle NONE OF THESE.
You do not need to explain your reasoning.

$h(-1) = -1$

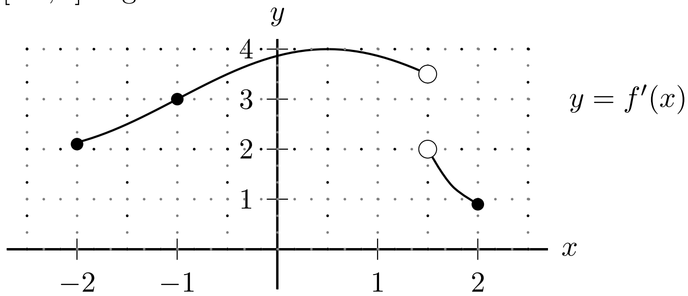
$h(-1) = 0$

$h(-1) = 1$

$h(-1) = 2$

NONE OF THESE

3. [8 points] Suppose $f(x)$ is a function that is continuous on the interval $[-2, 2]$. The graph of $f'(x)$ on the interval $[-2, 2]$ is given below.



a. [3 points] Let $L(x)$ be the local linearization of $f(x)$ at $x = -1$. Using the fact that $f(-1) = 4$, write a formula for $L(x)$.

Answer: $L(x) =$ ____________________

b. [2 points] Use your formula for $L(x)$ to approximate $f(-0.5)$.

Answer: $f(-0.5) \approx$ ____________________

c. [3 points] Is your answer from part (b) an overestimate or an underestimate of the actual value of $f(-0.5)$? Justify your answer.

Circle one: overestimate underestimate CANNOT BE DETERMINED

Justification:

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Answer: ____________________

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Answer: $T(205) \approx$ ____________________

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should not include the function names T or C.

Answer: $L(v) =$ ____________________

8. [14 points]

Suppose H is a differentiable function such that $H'(w)$ is also differentiable for $0 < w < 10$. Several values of $H(w)$ and of its first and second derivatives are given in the table on the right.

w	1	2	3	5	8
$H(w)$	6.3	5.4	5.2	4.8	0.7
$H'(w)$	−1.5	−0.4	−0.1	−0.6	−2.1
$H''(w)$	1.6	0.9	0	−0.8	−0.4

Assume that between each pair of consecutive values of w shown in the table, each of $H'(w)$ and $H''(w)$ is either always strictly decreasing or always strictly increasing. *Remember to show your work carefully.*

a. [3 points] Use an appropriate linear approximation to estimate $H(5.2)$.

Answer: $H(5.2) \approx$ ______________________

b. [5 points] Let $J(w)$ be the local linearization of H near $w = 2$, and let $K(w)$ be the local linearization of H near $w = 3$. Which of the following statements must be true? Circle all of the statements that must be true, or circle "NONE OF THESE".

$J(2) > H(2)$	$J(2.5) > H(2.5)$	$K(3.5) > H(3.5)$
$J(2) = H(2)$	$J(2.5) = H(2.5)$	$K(3.5) = H(3.5)$
$J(2) < H(2)$	$J(2.5) < H(2.5)$	$K(3.5) < H(3.5)$
$J'(2) > H'(2)$	$K(2.5) > H(2.5)$	$K'(3.5) > H'(3.5)$
$J'(2) = H'(2)$	$K(2.5) = H(2.5)$	$K'(3.5) = H'(3.5)$
$J'(2) < H'(2)$	$K(2.5) < H(2.5)$	$K'(3.5) < H'(3.5)$

NONE OF THESE

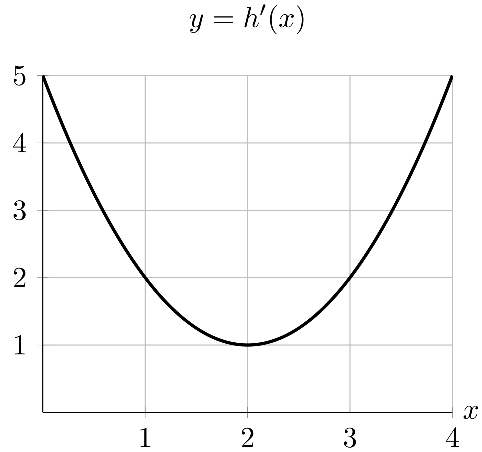
c. [3 points] Use the quadratic approximation of $H(w)$ at $w = 1$ to estimate $H(0.9)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Answer: $H(0.9) \approx$ ______________________

d. [3 points] Consider the function N defined by $N(w) = H(2w^2 - 10)$, and let $L(w)$ be the local linearization of $N(w)$ at $w = 3$. Find a formula for $L(w)$. Your answer should not include the function names N or H.

Answer: $L(w) =$ ______________________

9. [7 points] The graph of $h'(x)$ (the **derivative** of $h(x)$) is shown below.



a. [3 points] Find a formula for the tangent line approximation $L(x)$ to the function $h(x)$ near $x = 2$ if the point $(2, -3)$ lies on the graph of $y = h(x)$. Your answer should not include the letter h.

Answer: $L(x) =$ ________________________

b. [1 point] Use the tangent line approximation to the function $h(x)$ near $x = 2$ to approximate the value of $h(1.6)$.

Answer: $h(1.6)$ is approximately ______________

c. [3 points] Is your approximation in part **b** an overestimate or an underestimate? Circle your answer and give a justification of your answer.

OVERESTIMATE UNDERESTIMATE NOT ENOUGH INFORMATION

Justification:

10. [9 points] Consider the function h defined by $$h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80\ln\left(\frac{x}{2}\right) & \text{if } x \geq 2 \end{cases}$$

where A and B are constants.

a. [6 points] Find values of A and B so that h is differentiable.
Remember to show your work clearly.

Answer: $A =$ ____________________ and $B =$ ____________________

b. [3 points] Using the values of A and B you found in part **a.**, find the tangent line approximation for $h(x)$ near $x = 1$.

Answer: The tangent line approximation is given by $y =$ __________________________

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Answer: $\dfrac{dy}{dx} =$

11. [6 points] Let $h(x) = x^x$. For this problem, it may be helpful to know the following formulas:

$$h'(x) = x^x(\ln(x) + 1) \qquad \text{and} \qquad h''(x) = x^x\left(\frac{1}{x} + (\ln(x)+1)^2\right).$$

a. [2 points] Write a formula for $p(x)$, the local linearization of $h(x)$ near $x = 1$.

Answer: $p(x) =$ ____________________

b. [4 points] Write a formula for $u(x)$, the quadratic approximation of $h(x)$ at $x = 1$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Answer: $u(x) =$ ____________________