3. [9 points] Consider the curve $\mathcal{C}$ defined by

$$\cos(ax - y) + x^2 + y^2 = b$$

where a and b are positive constants.

a. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:
Implicit differentiation:

$$\frac{d}{dx}\left(\cos(ax - y) + x^2 + y^2\right) = \frac{d}{dx}(b)$$
$$\left(a - \frac{dy}{dx}\right)(-\sin(ax - y)) + 2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$-a\sin(ax - y) + \frac{dy}{dx}(\sin(ax - y)) + 2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(\sin(ax - y) + 2y) = a\sin(ax - y) - 2x$$
$$\frac{dy}{dx} = \frac{a\sin(ax - y) - 2x}{\sin(ax - y) + 2y}$$

Answer: $\frac{dy}{dx} = \frac{a\sin(ax - y) - 2x}{\sin(ax - y) + 2y}$

b. [1 point] Let $a = 1$ and $b = 9$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that one point.

$(3, 0)$ $\boxed{(2, 2)}$ $(1, -1)$ (π, π) $(0, -9)$

c. [3 points] With $a = 1$ and $b = 9$ as above, find an equation for the tangent line to the curve $\mathcal{C}$ at the point you chose in part **b.**.

Solution: To find the slope of the tangent line, we evaluate $\frac{dy}{dx}$ at the point $(2, 2)$ to find

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{\sin(2-2) - 2(2)}{\sin(2-2) + 2(2)} = -1.$$

Hence, the equation for the tangent line is $y = 2 - 1(x - 2) = -x + 4$.

Answer: $y = 2 - (x - 2)$ **(or** $-x + 4$**)**

4. [10 points]

a. Let $\mathcal{C}$ be the curve given by the equation

$$y\cos(2x) = y^3 + b,$$

where b is a constant. The curve $\mathcal{C}$ passes through the point $(0, 2)$.

i. [2 points] Find b.

Solution: Plugging in $(0, 2)$, we find that

$$2\cos(2 \cdot 0) = 2^3 + b$$

$$2 = 8 + b$$

$$b = -6.$$

Answer: $b =$ **-6**

ii. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:
Using implicit differentiation, and the product rule on the left-hand side,

$$-y\sin(2x) \cdot 2 + \frac{dy}{dx}\cos(2x) = 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}\cos(2x) - 3y^2\frac{dy}{dx} = 2y\sin(2x)$$

$$\frac{dy}{dx}\left(\cos(2x) - 3y^2\right) = 2y\sin(2x)$$

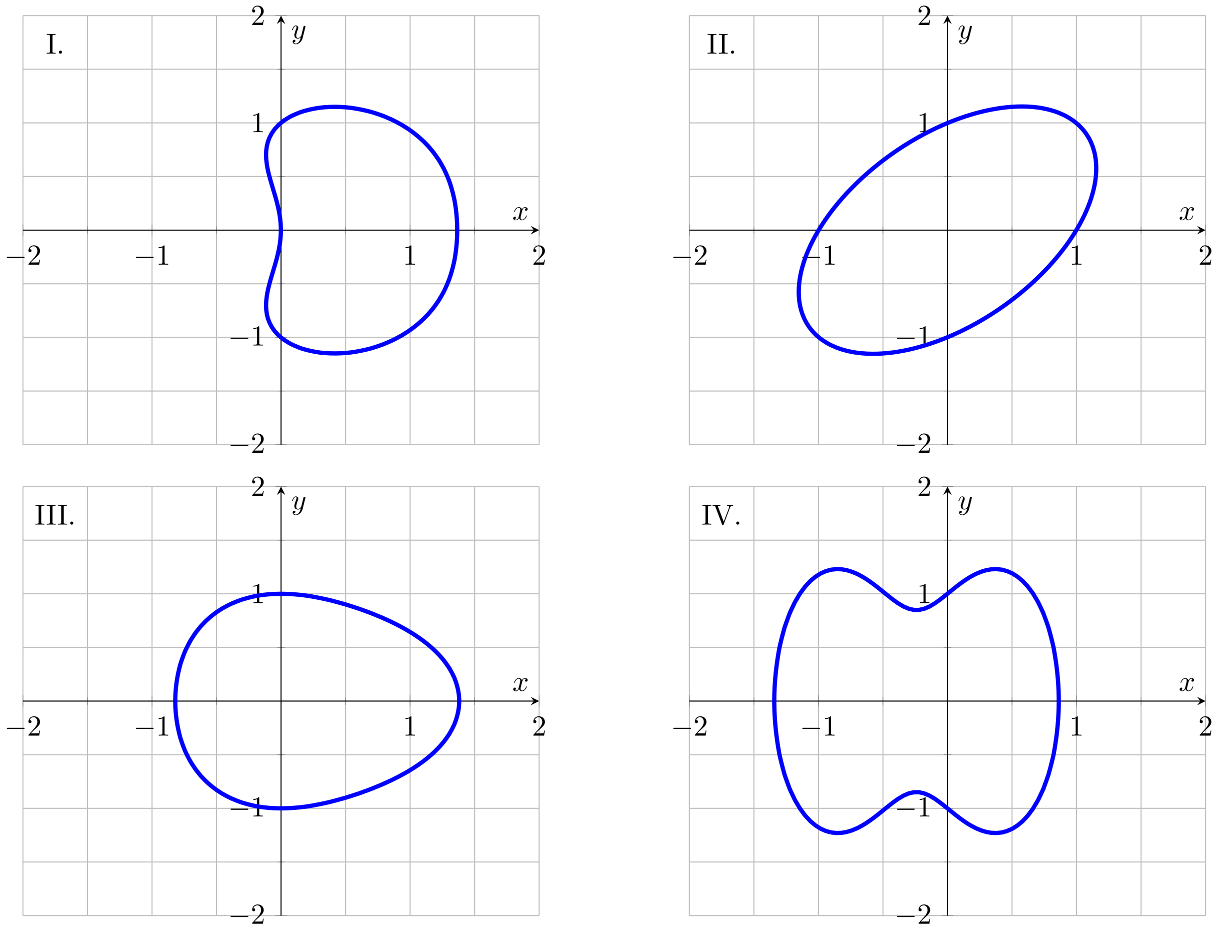
$$\frac{dy}{dx} = \frac{2y\sin(2x)}{\cos(2x) - 3y^2}$$

Answer: $\frac{dy}{dx} =$ $\frac{dy}{dx} = \frac{2y\sin(2x)}{\cos(2x) - 3y^2}$

b. [3 points] A different curve $\mathcal{R}$ passes through the point $(0, 1)$ and satisfies

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}.$$

One of the following graphs is the graph of $\mathcal{R}$. Which of the graphs is it? Write the numeral (I, II, III, or IV) of the graph you choose on the answer line at the bottom of this page.



Solution: We find that the slope at the given point $(0, 1)$ is $1/2$, so this rules out III. Finding that the slope at the point $(0, -1)$ must also be $1/2$, we conclude that II must be correct. (We could also have ruled out I and IV (and III) by noting that these graphs have vertical tangents when $y = 0$, but dy/dx is not undefined when $y = 0$.)

Answer: **II**

7. [10 points] For each real number k, there is a curve in the plane given by the equation

$$e^{y^2} = x^3 + k.$$

a. [4 points] Find $\frac{dy}{dx}$.

Solution: We have

$$2ye^{y^2}\frac{dy}{dx} = 3x^2,$$

so

$$\frac{dy}{dx} = \frac{3x^2}{2ye^{y^2}}$$

b. [3 points] Suppose that $k = 9$. There are two points on the curve where the tangent line is horizontal. Find the x and y coordinates of each one.

Solution: Horizontal tangent lines occur when the numerator of the derivative is zero, so in this case $x = 0$. To solve for the y-coordinate, we have

$$e^{y^2} = 9$$

so $y = \pm\sqrt{\ln(9)}$.

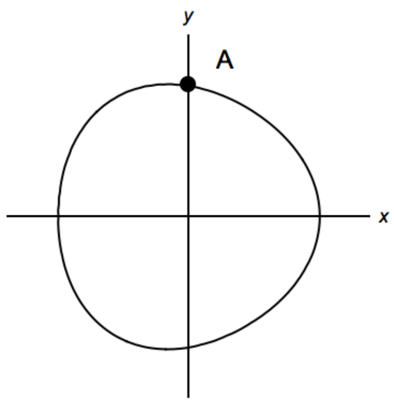
c. [3 points] Now suppose that $k = \frac{1}{2}$. How many points are there where the curve has a horizontal tangent line?

Solution: Again we get $x = 0$. Now if we try to solve for y we have

$$y^2 = \ln\left(\frac{1}{2}\right) < 0$$

and so there are no points where the curve has a horizontal tangent line.

8. [11 points] Let $\mathcal{C}$ be the curve given by the equation $81-(x^2+y^2)^2=2xy^2$. The graph of $\mathcal{C}$ is shown below.



a. [2 points] Find the coordinates (x,y) of the point A.

Solution: Since the point A lies at the intersection of the y-axis and the curve $\mathcal{C}$, then $x=0$ and y satisfies $81-(0^2+y^2)^2=2(0)xy^2$. Hence $y^4=81$ or $y=3$.

$A=(0,3)$

b. [6 points] Find $\dfrac{dy}{dx}$. Show all your computations step by step.

Solution:

$$\frac{d}{dx}\left(81-(x^2+y^2)^2\right)=\frac{d}{dx}\left(2xy^2\right)$$

$$-2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)=2y^2+4xy\frac{dy}{dx}$$

$$-4x(x^2+y^2)-4y(x^2+y^2)\frac{dy}{dx}=2y^2+4xy\frac{dy}{dx}$$

$$-4y(x^2+y^2)\frac{dy}{dx}-4xy\frac{dy}{dx}=2y^2+4x(x^2+y^2)$$

$$\frac{dy}{dx}=-\frac{2y^2+4x(x^2+y^2)}{4y(x^2+y^2)+4xy}$$

c. [3 points] Find an equation of the tangent line $L(x)$ to the graph of $\mathcal{C}$ at A. Show all your work.

Solution: The slope of $L(x)$ is

$$m=-\frac{2(3)^2+4(0)((0)^2+(3)^2)}{4(3)((0)^2+(3)^2)+4(0)(3)}=-\frac{18}{108}=-\frac{1}{6}.$$

Hence using the point A and the slope-intercept formula for the line $L(x)$, we get $L(x)=-\dfrac{1}{6}x+3$.

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}(\cos(ax) + by\ln(x)) &= \frac{d}{dx}(3 + y^3) \\ -a\sin(ax) + \frac{by}{x} + b\ln(x)\frac{dy}{dx} &= 3y^2\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)} \end{aligned}$$

Answer: $\frac{dy}{dx} = \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)}$