Implicit Functions

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1 Implicit Functions

Functions of the form $y = f(x)$ are called *explicit functions*. Equations involving x, y, such as $x^2 + y^2 = 4$, are said to give y as an *implicit function* of x.

Take $x^2 + y^2 = 4$ as an example. Since there are x-values which corresponds to two y-values, y is not a function of x on the whole circle. In fact we have

$$y = \pm\sqrt{4 - x^2}.$$

So y is a function of x in the top half, i.e., $y = \sqrt{4 - x^2}$ and it is also a function of x in the bottom half, i.e., $y = -\sqrt{4 - x^2}$.

2 Differentials of implicit functions

The rule to differentiate an implicit function is just differentiating the equation term by term. Again, take $x^2 + y^2 = 4$ as an exmaple. We will do

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= \frac{\mathrm{d}}{\mathrm{d}x}(4) \\ 2x + 2y \cdot y' &= 0 \\ 2y \cdot y' &= -2x \\ y' &= -\frac{x}{y}. \end{aligned}$$

Example 2.1. For $y = x^x$, let us apply natural log to both sides. So we have $\ln(y) = \ln(x^x) = x\ln(x)$. Now we can differentiate this implicit function

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\ln(y)) &= \frac{\mathrm{d}}{\mathrm{d}x}(x\ln(x)) \\ \frac{y'}{y} &= 1 + \ln(x) \\ y' &= y(1 + \ln(x)) = x^x(1 + \ln(x)) \end{aligned}$$

Example 2.2. If $y = \arcsin(x)$, then $x = \sin(y)$. So we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin(y)) \\ 1 &= \underline{\qquad\qquad} \\ y' &= \underline{\qquad\qquad} \end{aligned}$$

1. For each function below, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Assume that a, b are constants.

(a) $x^2 + y^2 = \sqrt{7}$

(b) $x^2 + y^3 = 8$

(c) $x^2 + xy - y^3 = xy^2$

(d) $x^2 y - 2y + 5 = 0$

(e) $\sqrt{x} + \sqrt{y} = 25$

(f) $\ln(x) + \ln(y^2) = 3$

(g) $x \ln(y) + y^3 = \ln(x)$

(h) $\sin(xy) = 2x + 5$

(i) $\mathrm{e}^{\cos(y)} = x^3 \arctan(y)$

(j) $\arctan(x^2 y) = xy^2$

(k) $\mathrm{e}^{x^2} + \ln(y) = 0$

(l) $\sin(ay) + \cos(bx) = xy$

3. [9 points] Consider the curve $\mathcal{C}$ defined by

$$\cos(ax - y) + x^2 + y^2 = b$$

where a and b are positive constants.

a. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show <u>every</u> step of your work clearly.

Answer: $\frac{dy}{dx} =$ ______________________________

b. [1 point] Let $a = 1$ and $b = 9$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that <u>one</u> point.

$(3, 0)$ $\quad$ $(2, 2)$ $\quad$ $(1, -1)$ $\quad$ (π, π) $\quad$ $(0, -9)$

c. [3 points] With $a = 1$ and $b = 9$ as above, find an equation for the tangent line to the curve $\mathcal{C}$ at the point you chose in part **b.**.

Answer: $y =$ ______________________________

4. [10 points]

a. Let $\mathcal{C}$ be the curve given by the equation

$$y\cos(2x) = y^3 + b,$$

where b is a constant. The curve $\mathcal{C}$ passes through the point $(0, 2)$.

i. [2 points] Find b.

Answer: $b =$ ____________

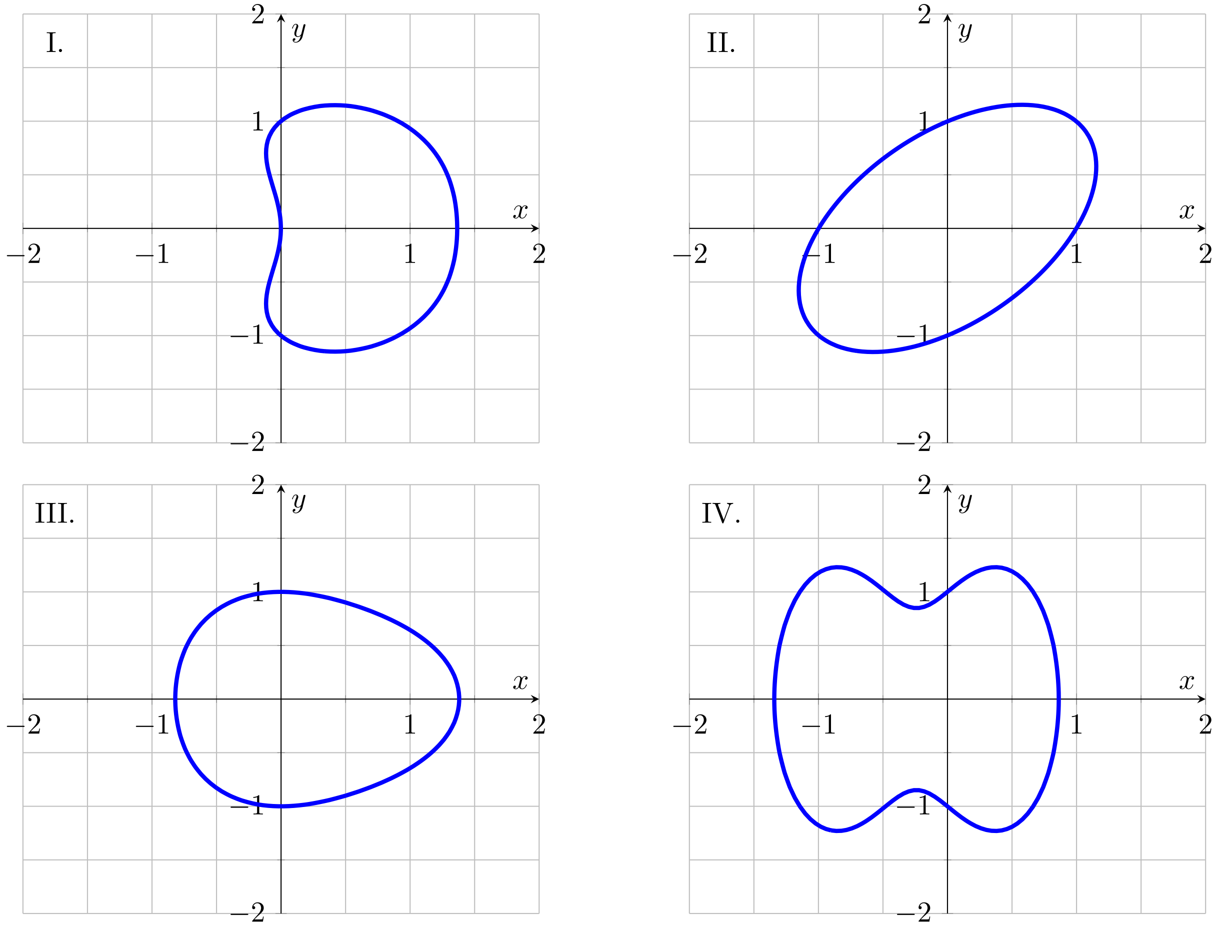
ii. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Answer: $\dfrac{dy}{dx} =$ ______________________________

b. [3 points] A different curve $\mathcal{R}$ passes through the point $(0, 1)$ and satisfies

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}.$$

One of the following graphs is the graph of $\mathcal{R}$. Which of the graphs is it? Write the numeral (I, II, III, or IV) of the graph you choose on the answer line at the bottom of this page.



Answer: ____________

7. [10 points] For each real number k, there is a curve in the plane given by the equation

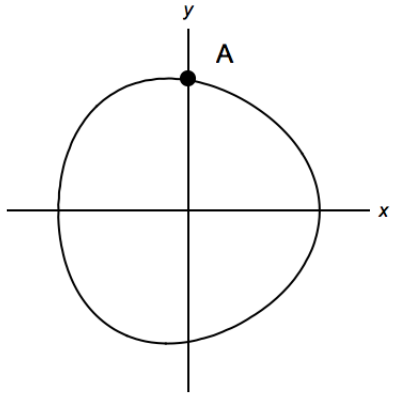
$$e^{y^2} = x^3 + k.$$

a. [4 points] Find $\frac{dy}{dx}$.

b. [3 points] Suppose that $k = 9$. There are two points on the curve where the tangent line is horizontal. Find the x and y coordinates of each one.

c. [3 points] Now suppose that $k = \frac{1}{2}$. How many points are there where the curve has a horizontal tangent line?

8. [11 points] Let $\mathcal{C}$ be the curve given by the equation $81 - (x^2 + y^2)^2 = 2xy^2$. The graph of $\mathcal{C}$ is shown below.



a. [2 points] Find the coordinates (x, y) of the point A.

$A =$ ______________

b. [6 points] Find $\frac{dy}{dx}$. Show all your computations step by step.

$\frac{dy}{dx} =$ ______________

c. [3 points] Find an equation of the tangent line $L(x)$ to the graph of $\mathcal{C}$ at A. Show all your work.

$L(x) =$ ______________

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Answer: $\dfrac{dy}{dx} =$ [ ]