The Chain Rule and Inverse Functions

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1 Derivative of an Inverse Function

1.1 The formula

Let $f(x)$ be an invertible function and write $f^{-1}(x)$ for its inverse. Let $g(x) = f(f^{-1}(x))$, then $g(x) = x$. So we immediately know that $g'(x) = 1$. On the other hand, by chain rule, we have

$$g'(x) = f'(f^{-1}(x)) \cdot (f^{-1}(x))'$$

So we equate these two to get

$$\begin{aligned} f'(f^{-1}(x)) \cdot (f^{-1}(x))' &= 1 \\ (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

2 Examples

2.1 Natural log

Recall that if $f(x) = \mathrm{e}^x$, then $f^{-1}(x) = \ln(x)$. Using the formula above, since $f'(x) = \mathrm{e}^x$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{\mathrm{e}^{\ln(x)}} = \frac{1}{x}$$

1. Find the derivative of following functions

   (a) $f(t) = \ln(t^2 + 1)$.

   (b) $f(x) = \ln(1 - x)$.

   (c) $f(x) = \ln(\mathrm{e}^{2x})$.

   (d) $f(\alpha) = \ln(\sin(\alpha))$.

2.2 Inverse trig functions

Let $f(x) = \sin(x)$. Then $f^{-1}(x) = \arcsin(x)$. Since $f'(x) = \cos(x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(x)) = \frac{1}{\cos(\arcsin(x))}$$

To find a nicer formula, we need to do some algebra. Write $\theta = \arcsin(x)$, then $x = \sin(\theta)$. Recall that $\sin^2(\theta) + \cos^2(\theta) = 1$. So $\cos(\theta) = \sqrt{1 - x^2}$. Then

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1 - x^2}}$$

We can use similar methods to find the derivative of $\arccos(x)$. But here is a quicker way. Notice that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$. So

$$(\arccos(x))' = \left(\frac{\pi}{2} - \arcsin(x)\right)' = -(\arcsin(x))' = -\frac{1}{\sqrt{1 - x^2}}$$

Next let us find the formula for $\arctan(x)$. Note that $(\tan(x))' = \frac{1}{\cos^2(x)}$. So

$$(\arctan(x))' = \frac{1}{1/\cos^2(\arctan(x))} = \cos^2(\arctan(x)).$$

Again we want to find out $\cos(\arctan(x))$. Let $\theta = \arctan(x)$. Then $\tan(\theta) = x$, which is $\frac{\sin(\theta)}{\cos(\theta)} = x$. This equation, together with $\cos^2(\theta) + \sin^2(\theta) = 1$, shows that $\cos(\theta) = \frac{1}{\sqrt{1 + x^2}}$. Hence

$$(\arctan(x))' = \frac{1}{1 + x^2}$$

1. Find derivatives of following functions

   (a) $y = \arcsin(x + 1)$.

   (b) $f(x) = \arctan(3x)$.

   (c) $f(y) = \arcsin(y^2)$.

   (d) $s(x) = \arctan(2 - x)$.

   (e) $g(t) = \mathrm{e}^{\arctan(3t^2)}$.

   (f) $j(x) = \cos(\sin^{-1}(x))$.

   (g) $f(x) = \cos(\arctan(3x))$.

   (h) $f(x) = \ln(\sin(x) + \cos(x))$

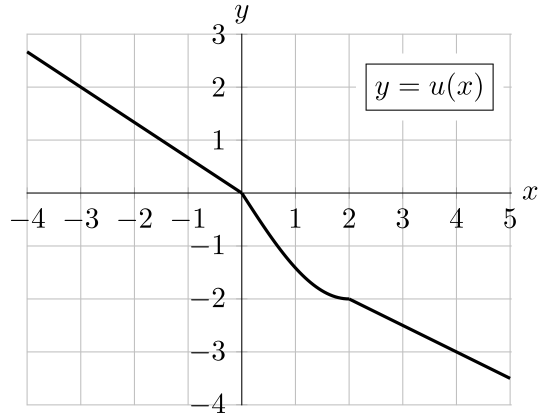
3. [9 points] This problem concerns the function $f(x) = -x - 3e^{4x}$.

a. [3 points] Show that the function f is invertible.

b. [2 points] Find $f^{-1}(-3)$. You do not need to show any work.

c. [4 points] Evaluate $(f^{-1})'(-3)$. Show all of your work.

4. [13 points]



The function $u(x)$ is defined and invertible on $(-\infty, \infty)$. A portion of its graph is shown to the right.

Note that:

- $u(x) = -2\sin\left(\frac{\pi}{4}x\right)$ on $[0, 2]$, and
- $u(x)$ is linear on the intervals $(-4, 0)$ and $(2, 5)$.

a. [11 points] Evaluate each of the following quantities **exactly**, or write DNE if the value does not exist. You do not need to show work, but limited partial credit may be awarded for work shown. Your answers should not contain the letter u, but do not need to be fully simplified.

i. [2 points] Find $(u^{-1})'(-3)$.

Answer: $(u^{-1})'(-3) =$ ______________________

ii. [2 points] Let $v(x) = u(-1 - x)$. Find $v'(-1)$.

Answer: $v'(-1) =$ ______________________

iii. [3 points] Let $w(x) = \dfrac{x}{2^{u(x)}}$. Find $w'(-3)$.

Answer: $w'(-3) =$ ______________________

iv. [4 points] Let $z(x) = \ln(2x + 1)u(x)$. Find $z'(1)$.

Answer: $z'(1) =$ ______________________

b. [2 points] At $x = 7$, the tangent line to $u(x)$ is given by $y = -5 - 2(x - 7)$. Find an equation for the tangent line to $u^{-1}(x)$ at $x = -5$.

Answer: ______________________