Trigonometric Rule and Chain Rule

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1 Trigonometric Rule

Theorem 1.1 (Trig Rule)**.** *For x in radians,*

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$$

Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and we have quotient rule, therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) &= \frac{(\sin(x))' \cos(x) - (\cos(x))' \sin(x)}{\cos^2(x)} \\ &= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)}\end{aligned}$$

1. Find the derivative of following trigonometric functions.
   (a) $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$
   (b) $\sec(x) = \dfrac{1}{\cos(x)}$
   (c) $\csc(x) = \dfrac{1}{\sin(x)}$

2 The MOST IMPORTANT Rule - Chain Rule

Theorem 2.1 (Chain Rule)**.** *If $z = f(g(x))$ is a composition function of f and g, then the derivative of z with respect to x is*

$$z' = f'(g(x)) \cdot g'(x)$$

If we write $y = g(x)$, then $z = f(y)$. With Leibniz notation, we have

$$\frac{\mathrm{d}z}{\mathrm{d}y} = f'(y) = f'(g(x))$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = g'(x)$$

So chain rule can be written as

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

Let us go through some examples:

Example 2.2. Using chain rule to find the derivative of e^{2x}, we note that if $f(x) = \mathrm{e}^x$, $g(x) =$ ______________, then $\mathrm{e}^{2x} = f(g(x))$. By chain rule, we have

$$(\mathrm{e}^{2x})' = f'(g(x)) \cdot g'(x) = \mathrm{e}^{g(x)} \cdot g'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

1. Find the derivative of following functions

(a) $g(x) = \mathrm{e}^{\pi x}$

(b) $B = 15\mathrm{e}^{0.20t}$

(c) $f(\theta) = 2^{-\theta}$

(d) $g(x) = 3^{2x+7}$

(e) $w = \mathrm{e}^{\sqrt{s}}$

(f) $y = \pi^{x+2}$

Example 2.3. To find the derivative of $(x+1)^{10}$, we can either expand it out and differentiate term by term, or use chain rule by setting $f(x) =$ ______________ and $g(x) =$ ______________, then $(x+1)^{10} = f(g(x))$. So

$$\left((x+1)^{10}\right)' = f'(g(x)) \cdot g'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

1. Find the derivative of following functions

(a) $g(x) = (4x^2 + 1)^7$

(b) $s(t) = (3t^2 + 4t + 1)^3$

(c) $w(r) = \sqrt{r^4 + 1}$

(d) $k(x) = (x^3 + \mathrm{e}^x)^4$

Example 2.4. Let $h(x) = x$, then $h'(x) = 1$. On the other hand, if we write $h(x) = \mathrm{e}^{\ln(x)}$, then by chain rule, we have $h'(x) = f'(g(x)) \cdot g'(x)$ where $f(x) = \mathrm{e}^x$ and $g(x) = \ln(x)$. So

$$1 = \mathrm{e}^{g(x)} \cdot g'(x) \Rightarrow g'(x) = \frac{1}{\mathrm{e}^{g(x)}} = \underline{\hspace{3cm}}$$

Example 2.5. 1. Let $p(x) = \frac{1}{x}$ and $h(x) = p(g(x))$, then (by chain rule),

$$h'(x) = \underline{\hspace{3cm}} \quad \text{a formula involving } g \text{ and } g'$$

2. Let $j(x) = \frac{f(x)}{g(x)} = f(x) \cdot p(g(x))$, use the result above and product rule to find

$$j'(x) = \underline{\hspace{3cm}} \quad \text{a formula involving } f, g, f' \text{ and } g'$$

1. Find the derivative of following functions

(a) $r(\theta) = \sin(\theta)\cos(\theta)$

(b) $y = t\mathrm{e}^{-t^2}$

(c) $z = \theta \mathrm{e}^{\cos(\theta)}$

(d) $y = \frac{\sqrt{z}}{2^z}$

(e) $w = \sqrt{(x^2 \cdot 5^x)^3}$

(f) $g(\theta) = \sin(\tan(\theta))$

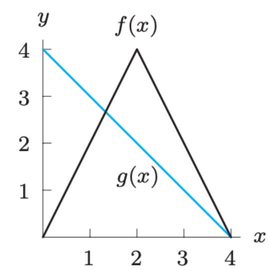
(g) $y = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} (= \tanh(x))$

(h) $y = \left(x^2 + 5\right)^3 \left(3x^3 - 2\right)^2$

(i) $y = \sin(\sin(\theta) + \cos(\theta))$

(j) $f(t) = 2\mathrm{e}^{-2\mathrm{e}^{2t}}$

2. Use figure below to estimate the derivatives, or state why the derivative does not exist. Note that the graph of $f(x)$ has a sharp corner.



(a) Let $h(x) = f(g(x))$. Find $h'(1), h'(2)$ and $h'(3)$.

(b) Let $u(x) = g(f(x))$. Find $u'(1), u'(2)$ and $u'(3)$.

(c) Let $v(x) = f(f(x))$. Find $v'(1), v'(2)$ and $v'(3)$.

(d) Let $w(x) = g(g(x))$. Find $w'(1), w'(2)$ and $w'(3)$.

3. Find the equation of the line tangent to $y = f(x)$ at $x = 1$ where $f(x) = 6e^{5x} + e^{-x^2}$.

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Answer: $g'(1) =$ ____________________

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Answer: $h'(\ln 2) =$ ____________________

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Answer: $j'(-2) =$ ____________________

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

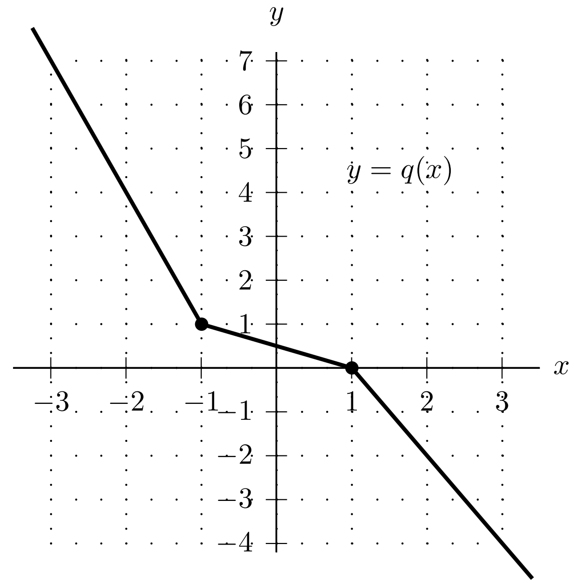
Answer: $k'(1) =$ ____________________

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Answer: $\ell'(0) =$ ____________________

1. [11 points]

Shown to the right is the graph of an invertible piecewise linear function $q(x)$. Note that the graph passes through the points $(-3, 7)$, $(-1, 1)$, $(1, 0)$, and $(3, -4)$.



You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

Find the exact value of each of the quantities below. If there is not enough information provided to find the value, write "NOT ENOUGH INFO". If the value does not exist, write "DOES NOT EXIST".

b. [3 points] Let $w(x) = \dfrac{x}{q(x+1)}$. Find $w'(-2)$.

Answer: $w'(-2) =$ ____________________

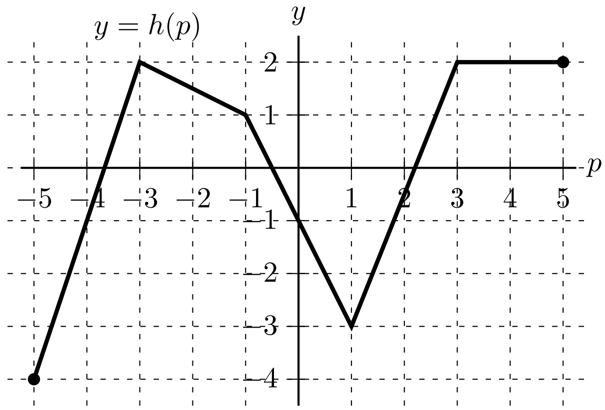
c. [3 points] Let $v(x) = xq(\sin x)$. Find $v'(\pi)$.

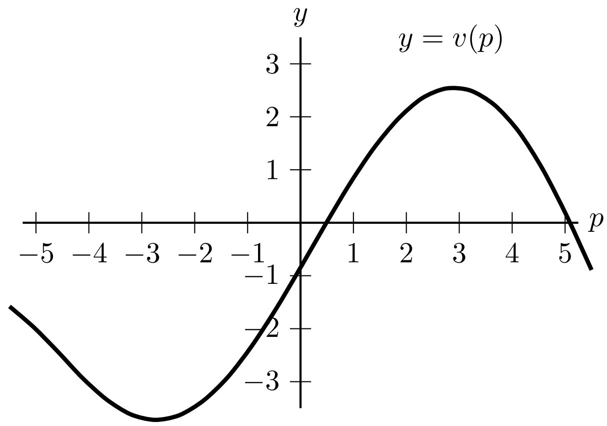
Answer: $v'(\pi) =$ ____________________

d. [3 points] Let $j(x) = \ln(q(2x))$. Find $j'(-1)$.

Answer: $j'(-1) =$ ____________________

1. [12 points] The graphs of two functions, $h(p)$ and $v(p)$, are shown below.





The following questions concern the functions B, W, and Q defined as follows:

$$B(p) = \frac{h(2p)}{h(4p)}, \qquad W(p) = h(h(p)), \quad \text{and} \quad Q(p) = e^{-v(p)}.$$

Assume that the first and second derivatives of $v(p)$ are defined everywhere, i.e. that both v and v' are differentiable on $(-\infty, \infty)$. Note that the graph of $h(p)$ consists of line segments whose endpoints have integer (whole number) coordinates. Find the exact value of each of the quantities in **a.** and **b.** below. If the value does not exist, write DOES NOT EXIST. *Remember to show your work carefully.*

a. [4 points] $B'(-1)$

Answer: $B'(-1) =$ ________________________

b. [4 points] $W'(2)$

Answer: $W'(2) =$ ________________________

c. [4 points] On the interval $-2 < p < 2$, is $Q(p)$ always increasing, always decreasing, or neither? Show your work and explain your reasoning.

1. [13 points] Some values of the twice differentiable function $f(x)$ and of its first and second derivative are given by the following table

x	0	1	2	4	5	6	7
$f(x)$	1			4	4.3	5	
$f'(x)$			8		0.25	0.6	2
$f''(x)$	4				0.1	0.2	

Suppose the function $f(x)$ is defined and invertible for $-\infty < x < \infty$. In the following questions, you will find some of the missing values using the information given. If there is not enough information given to answer the question, write "NEI". Show your work.

a. [4 points] The function $a(x) = \ln(1 + f(x))$ satisfies $a'(2) = 2$. Find $f(2)$.

Answer: $f(2) =$ ______________

b. [3 points] Let $b(x) = f(x)f'(x)$ and $b'(0) = 4$. Find $f'(0)$.

Answers: $f'(0) =$ ______________

The problem continues on the next page.

For your convenience, the table with some values of $f(x)$, $f'(x)$, and $f''(x)$ has been reproduced below.

x	0	1	2	4	5	6	7
$f(x)$	1			4	4.3	5	
$f'(x)$			8		0.25	0.6	2
$f''(x)$	4				0.1	0.2	

Suppose the function $f(x)$ is defined and invertible for $-\infty < x < \infty$. Answer the following questions. If there is not enough information given to answer the question, write "NEI". Show your work.

c. [3 points] The quadratic approximation $Q(x)$ of the function $f(x)$ at $x = 1$ is $Q(x) = \frac{1}{2}x + \frac{3}{2}$. Find $f(1)$, $f'(1)$, and $f''(1)$.

Answers: $f(1) =$ ____________, $f'(1) =$ ____________, $f''(1) =$ ____________