1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Solution: $g'(x) = 3f'(x)$, so $g'(1) = 3 \cdot 0.5 = 1.5$

Answer: $g'(1) =$ 1.5

b. [2 points] Find $(f^{-1})'(2)$.

Solution: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, so $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(3)} = \dfrac{1}{0.1} = 10.$

Answer: $(f^{-1})'(2) =$ 10

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Solution: $h'(x) = f'(e^x) \cdot e^x$, so $h'(\ln 2) = f'(e^{\ln 2}) \cdot e^{\ln 2} = f'(2) \cdot 2 = 0.3 \cdot 2 = 0.6.$

Answer: $h'(\ln 2) =$ 0.6

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Solution: $j'(x) = e^{f(x)} \cdot f'(x)$, so $j'(-2) = e^{f(-2)} \cdot f'(-2) = e^{-4} \cdot 3.$

Answer: $j'(-2) =$ $3e^{-4}$

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Solution: $k'(x) = f'(x)f(x-2) + f(x)f'(x-2)$, so
$k'(1) = f'(1)f(1-2) + f(1)f'(1-2) = f'(1)f(-1) + f(1)f'(-1) = 0.5 \cdot (-1.2) + 1.4 \cdot 2 =$
$-0.6 + 2.8 = 2.2.$

Answer: $k'(1) =$ 2.2

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

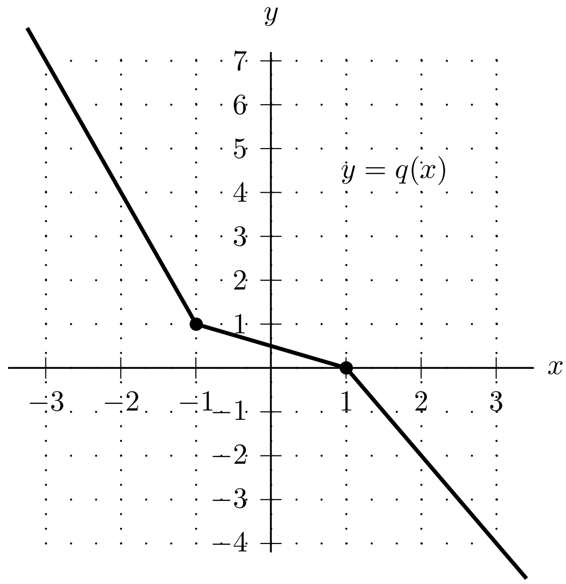
Solution: $\ell'(x) = \dfrac{f'(x)f(x+3) - f'(x+3)f(x)}{(f(x+3))^2}$, so

$$\ell'(0) = \frac{f'(0)f(3) - f'(3)f(0)}{(f(3))^2} = \frac{1.2 \cdot 2 - 0.5 \cdot 0.1}{2^2} = \frac{2.4 - 0.05}{4} = \frac{2.35}{4} = 0.5875.$$

Answer: $\ell'(0) =$ 0.5875

1. [11 points]

Shown to the right is the graph of an invertible piecewise linear function $q(x)$. Note that the graph passes through the points $(-3, 7)$, $(-1, 1)$, $(1, 0)$, and $(3, -4)$.



You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

Find the exact value of each of the quantities below. If there is not enough information provided to find the value, write "NOT ENOUGH INFO". If the value does not exist, write "DOES NOT EXIST".

a. [2 points] Let $r(x) = q^{-1}(x)$. Find $r'(2)$.

Solution: $r'(x) = \dfrac{1}{q'(q^{-1}(x))}$ so $r'(2) = \dfrac{1}{q'(q^{-1}(2))} = \dfrac{1}{-3} = -\dfrac{1}{3}$.

Answer: $r'(2) =$ $-\dfrac{1}{3}$

b. [3 points] Let $w(x) = \dfrac{x}{q(x+1)}$. Find $w'(-2)$.

Solution: By the quotient and chain rules, $w'(x) = \frac{q(x+1) - xq'(x+1)}{(q(x+1))^2}$ (where these quantities are defined). q' is not differentiable at $x = -1$, so $q'(x+1)$ is not defined at $x = -2$. (If $w'(-2)$ were to exist, then since $q(x+1) = \frac{w(x)}{x}$, we would have $q'(-1) = q'(-2+1) = \frac{(-2)w'(-2) - w(-2)}{(-2)^2}$.)

Answer: $w'(-2) =$ DOES NOT EXIST

c. [3 points] Let $v(x) = xq(\sin x)$. Find $v'(\pi)$.

Solution: By the product and chain rules we have $v'(x) = xq'(\sin x)\cos x + q(\sin x)$. So $v'(\pi) = \pi q'(\sin \pi)\cos \pi + q(\sin \pi) = \pi q'(0)(-1) + q(0) = \pi(-1/2)(-1) + (1/2) = \frac{\pi+1}{2}$.

Answer: $v'(\pi) =$ $\dfrac{\pi + 1}{2}$

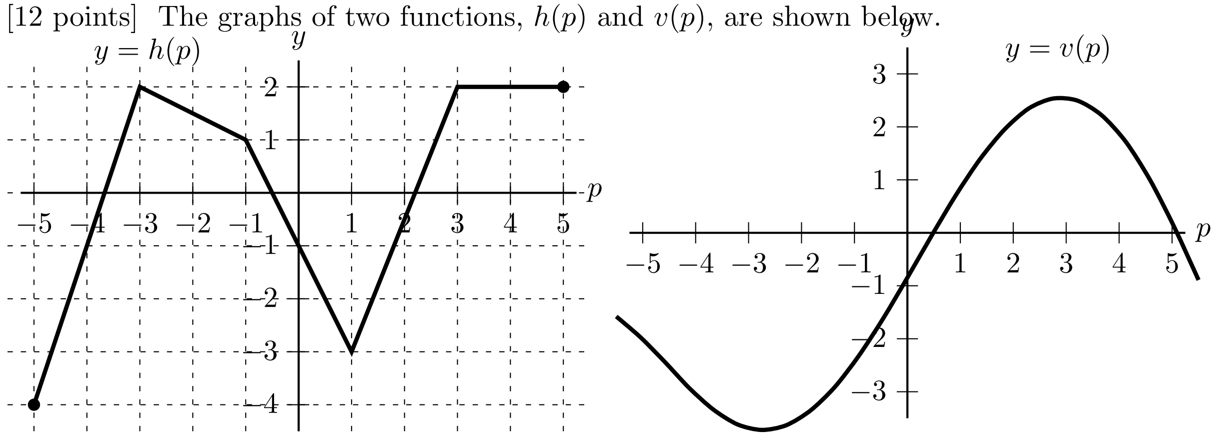
d. [3 points] Let $j(x) = \ln(q(2x))$. Find $j'(-1)$.

Solution: By the chain rule, we have

$$j'(x) = \frac{1}{q(2x)} \cdot q'(2x) \cdot 2 = \frac{2q'(2x)}{q(2x)} \quad \text{so} \quad j'(-1) = \frac{2q'(-2)}{q(-2)} = \frac{2(-3)}{4} = -\frac{3}{2}.$$

Answer: $j'(-1) =$ $-\dfrac{3}{2}$

1. [12 points] The graphs of two functions, $h(p)$ and $v(p)$, are shown below.



The following questions concern the functions B, W, and Q defined as follows:

$$B(p) = \frac{h(2p)}{h(4p)}, \qquad W(p) = h(h(p)), \quad \text{and} \quad Q(p) = e^{-v(p)}.$$

Assume that the first and second derivatives of $v(p)$ are defined everywhere, i.e. that both v and v' are differentiable on $(-\infty, \infty)$. Note that the graph of $h(p)$ consists of line segments whose endpoints have integer (whole number) coordinates. Find the exact value of each of the quantities in **a.** and **b.** below. If the value does not exist, write DOES NOT EXIST. *Remember to show your work carefully.*

a. [4 points] $B'(-1)$

Solution: Applying the quotient and chain rules, we have

$$B'(p) = \frac{2h'(2p)h(4p) - 4h'(4p)h(2p)}{(h(4p))^2}.$$

So $\quad B'(-1) = \dfrac{2h'(-2)h(-4) - 4h'(-4)h(-2)}{h(-4)^2} = \dfrac{2(-\frac{1}{2})(-1) - 4(3)(\frac{3}{2})}{(-1)^2} = -17.$

Answer: $B'(-1) =$ -17

b. [4 points] $W'(2)$

Solution: By the chain rule, $W'(p) = h'(h(p))h'(p)$, so

$$W'(2) = h'(h(2))h'(2) = h'(-\frac{1}{2})h'(2) = (-2)(\frac{5}{2}) = -5.$$

Answer: $W'(2) =$ -5

c. [4 points] On the interval $-2 < p < 2$, is $Q(p)$ always increasing, always decreasing, or neither? Show your work and explain your reasoning.

Solution: By the chain rule, $Q'(p) = -v'(p)e^{-v(p)}$. Since $e^x > 0$ for all x, we know that $e^{-v(p)}$ is always positive. On the interval $-2 < p < 2$, we can see that $v(p)$ is increasing and never has a horizontal tangent line, which means that $v'(p) > 0$ on this interval. Thus $Q'(p) = -v'(p)e^{-v(p)}$ is always negative on that interval, which means that $Q(p)$ is always decreasing on this interval.

1. [13 points] Some values of the twice differentiable function $f(x)$ and of its first and second derivative are given by the following table

x	0	1	2	4	5	6	7
$f(x)$	1			4	4.3	5	
$f'(x)$			8		0.25	0.6	2
$f''(x)$	4				0.1	0.2	

Suppose the function $f(x)$ is defined and invertible for $-\infty < x < \infty$. In the following questions, you will find some of the missing values using the information given. If there is not enough information given to answer the question, write "NEI". Show your work.

a. [4 points] The function $a(x) = \ln(1 + f(x))$ satisfies $a'(2) = 2$. Find $f(2)$.

Solution:

$$\begin{aligned} a'(x) &= \frac{1}{1+f(x)} f'(x) \\ 2 &= \frac{8}{1+f(2)} \\ 8 &= 2 + 2f(2) \\ f(2) &= 3 \end{aligned}$$

Answer: $f(2) = 3$.

b. [3 points] Let $b(x) = f(x)f'(x)$ and $b'(0) = 4$. Find $f'(0)$.

Solution:

$$\begin{aligned} b'(x) &= (f'(x))^2 + f(x)f''(x) \\ 4 &= (f'(0))^2 + f(0)f''(0) \\ 4 &= (f'(0))^2 + 4 \\ f'(0) &= 0. \end{aligned}$$

Answers: $f'(0) = 0$.

c. [3 points] The quadratic approximation $Q(x)$ of the function $f(x)$ at $x = 1$ is

$Q(x) = \frac{1}{2}x + \frac{3}{2}$. Find $f(1)$, $f'(1)$, and $f''(1)$.

Solution: **Answers:** $f(1) = 2$, $f'(1) = \frac{1}{2}$, $f''(1) = 0$

d. [3 points] Let $h(x) = f^{-1}(x)$. Find the value of $h'(5)$.

Solution: $h'(x) = \frac{1}{f'(f^{-1}(x))}$, then $h'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(6)} = \dfrac{1}{0.6} = \dfrac{5}{3}$.

Answer: $h'(5) = \dfrac{5}{3}$