Product/Quotient Rule, Exponential Rule

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1 Exponential Rule

Fact 1.1. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\right) = \mathrm{e}^x$$

For general exponential function $y = a^x$. Let us write

$$\begin{aligned}\frac{\mathrm{d}a^x}{\mathrm{d}x} &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h - 1}{h}\\ &= a^x \ln(a) \lim_{h\to 0}\frac{\mathrm{e}^{h\ln(a)} - 1}{h\ln(a)}\\ &= a^x \ln(a)\end{aligned}$$

So

Theorem 1.2 (Exponential Rule)**.** *If $y = a^x$, then*

$$y' = a^x \ln(a)$$

1. Find the derivatives of functions below

   (a) $z = (\ln 4)\mathrm{e}^x$

   (b) $y = \dfrac{3^x}{3} + \dfrac{33}{\sqrt{x}}$

   (c) $y = 2^x + \dfrac{2}{x^3}$

   (d) $z = (\ln 4)4^x$

   (e) $f(t) = (\ln 3)^t$

   (f) $y = 5\cdot 5^t + 6\cdot 6^t$

   (g) $f(x) = \mathrm{e}^2 + x^{\mathrm{e}}$

   (h) $f(\theta) = \mathrm{e}^{k\theta} - 1$

   (i) $y(x) = a^x + x^a$

2 The Product Rule

Suppose that we want to calculate the derivative of $f(x)g(x)$. The derivative is defined using limit

$$\frac{\mathrm{d}[f(x)g(x)]}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We have

$$\begin{aligned}
\frac{\mathrm{d}[f(x)g(x)]}{\mathrm{d}x} &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right) + g(x)\left(\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\right) \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Theorem 2.1 (Product Rule)**.** *If $y = f(x)$ and $z = g(x)$ are differentiable, then*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \textit{or} \quad \frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}v + \frac{\mathrm{d}v}{\mathrm{d}x}u$$

3 The Quotient Rule

Suppose that we want to calculate the derivative of $Q(x) = f(x)/g(x)$. Let us write $f(x) = g(x)Q(x)$. Then by Theorem 2.1, we have $f'(x) = Q'(x)g(x) + Q(x)g'(x)$. Solve for $Q'(x)$ we have

$$\begin{aligned}
Q'(x) &= \frac{f'(x) - Q(x)g'(x)}{g(x)} \\
&= \frac{f'(x) - (f(x)/g(x))g'(x)}{g(x)} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}
\end{aligned}$$

Theorem 3.1 (Quotient Rule)**.** *If $y = f(x)$ and $z = g(x)$ are differentiable, then*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \quad \textit{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

4 Questions

1. Find derivatives of following functions

   (a) $y = \sqrt{x} \cdot 2^x$

   (b) $y = (t^2 + 3)e^t$

   (c) $f(x) = (x^2 - \sqrt{x})3^x$

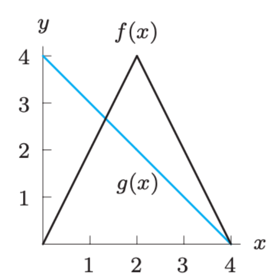
   (d) $g(t) = \dfrac{t-4}{t+4}$

   (e) $g(t) = \dfrac{4}{3+\sqrt{t}}$

   (f) $f(x) = \dfrac{ax+b}{cx+k}$

   (g) $f(x) = (2 - 4x - 3x^2)(6x^e - 3\pi)$

2. Use figure below to estimate the derivatives, or state why the derivative does not exist. Note that the graph of $f(x)$ has a sharp corner.



   (a) Let $h(x) = f(x) \cdot g(x)$. Find $h'(1), h'(2)$ and $h'(3)$.

   (b) Let $k(x) = f(x)/g(x)$. Find $k'(1), k'(2)$ and $k'(3)$.

   (c) Let $j(x) = g(x)/f(x)$. Find $j'(1), j'(2)$ and $j'(3)$.

3. Suppose that f and g are differentiable functions with the values shown in the following table. For each of the following functions h, find $h'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	5	-2

(a) $h(x) = f(x) + g(x)$

(b) $h(x) = f(x)g(x)$

(c) $h(x) = \dfrac{f(x)}{g(x)}$

4. (a) Differentiate $f(t) = \mathrm{e}^{-t}$ by writing it as $f(t) = \dfrac{1}{\mathrm{e}^t}$.

(b) Differentiate $f(t) = \mathrm{e}^{2t}$ by writing it as $f(t) = \mathrm{e}^t \cdot \mathrm{e}^t$.

(c) Differentiate $f(t) = \mathrm{e}^{3t}$ by writing it as $f(t) = \mathrm{e}^t \cdot \mathrm{e}^{2t}$.

5. (a) For what intervals is $f(x) = x\mathrm{e}^x$ concave up?

(b) For what intervals is $g(x) = \dfrac{1}{x^2+1}$ concave down?

6. Find and simplify $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f(x)g(x)\right)$