Powers and Polynomials

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1 Rules of taking derivatives

Theorem 1.1 (Derivative of a Constant Multiple)**.** *If f is differentiable and c is a constant, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(cf(x)\right) = cf'(x)$$

Theorem 1.2 (Derivative of Sum and Difference)**.** *If f and g are differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

2 Power rule and polynomials

Recall that we learned power rule for positive integer powers, in fact, this is true for any real number powers.

Theorem 2.1 (Power Rule)**.** *For any constant real number n,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$$

For example,

- the derivative of $y = x^{1/3}$ is $y = (1/3)x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.
- the derivative of $y = \sqrt{\frac{1}{x^5}}$: We first re-write the function as $y = \sqrt{x^{-5}} = x^{-5/2}$. Then $y' = (-5/2)x^{(-5/2)-1} = \frac{-5}{2}x^{-7/2}$.

Practice:

1. Find the derivative of each function below.
   (a) $y = -x^{-11}$
   (b) $y = x^{3.2}$
   (c) $y = \frac{1}{t^5}$
   (d) $y = \sqrt[4]{x}$

Now we know how to differentiate powers, constant multiples, and sums, we can differentiate any polynomial. For example, to find the derivative of $y = t^3 - 3t^2 + 8t - 4$. We work term by term

$$\begin{aligned} t^3 &\rightsquigarrow 3t^2 \\ -3t^2 &\rightsquigarrow -6t \\ 8t &\rightsquigarrow 8 \\ -4 &\rightsquigarrow 0 \end{aligned}$$

So $y' = 3t^2 - 6t + 8$.

1. Find the derivative of each function below.
   (a) $y = 4x^{3/2} - 5x^{2/3}$.
   (b) $y = 17x + 24x^{1/2}$.
   (c) $y = z^2 + \frac{1}{2z}$.
   (d) $y = \frac{x^2+\sqrt{x}+1}{x^{3/2}}$

3 Tangent Line Approximations

For any function $f(x)$ and for x near a, we have

$$f(x) \approx f(a) + f'(a)(x - a)$$

4 Questions

1. If $p(t) = t^3 + 2t^2 - t + 4$, find $\mathrm{d}^2p/\mathrm{d}t^2$ and $\mathrm{d}^3p/\mathrm{d}t^3$.

2. Find the equation of the line tangent to $y = x^2 + 3x - 5$ at $x = 2$.

3. For what values of x is the function $y = x^5 - 5x$ both incresing and concave up?

4. A rubber balloon contains neon. As the air pressure P (in atmospheres) outside the balloon increases, the volume of gas V (in liters) in the balloon decreases according to $V = f(P) = \frac{25}{P}$.
   (a) Evaluate and interpret $f(2)$, including unites.
   (b) Evaluate and interpret $f'(2)$, including unites.
   (c) Assuming that the pressure increases at a constant rate, does the volume of the balloon decrease faster when the pressure is 1 atmosphere or when the pressue is 2 atmosphere? Justify your answer.

5. The period T of a pendulum is given in terms of its length l, by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

   where g is the acceleration due to gravity (a constant).
   (a) Find $\frac{\mathrm{d}T}{\mathrm{d}l}$.
   (b) What is the sign of $\mathrm{d}T/\mathrm{d}l$? What does this tell you about the period of pendulums?