Differentiability

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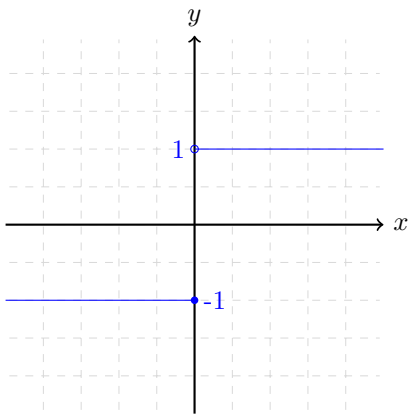
1 Definitions and Examples

Definition 1.1. A function f is *differentiable* at x if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{exists.}$$

A function fails to be differentiable at a point if

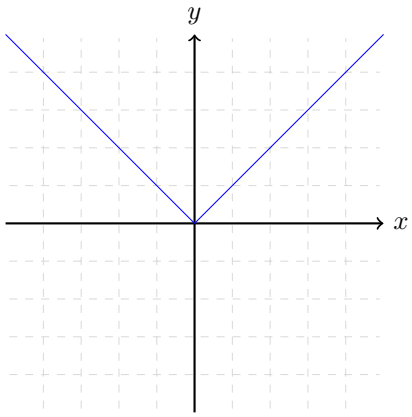
(1) The function is not continuous at the point. For example, $y=\begin{cases}1 & x>0\\ -1 & x\leqslant 0\end{cases}$ is not differentiable at $x=0$.



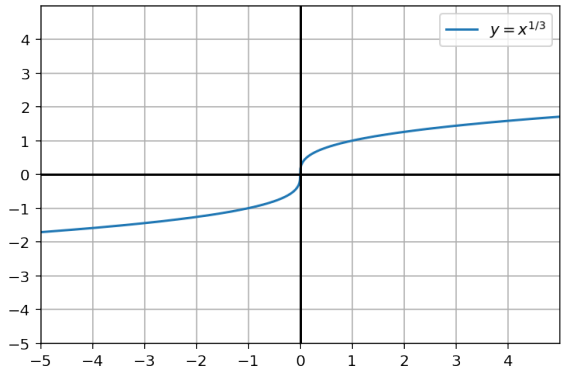
(2) The graph has a sharp corner at that point. Mathematically, this is saying that

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \neq \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

For examples, $y = |x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$ is not differentiable at $x = 0$.



(3) The graph has a vertical tangent line at x. For example, $y = x^{1/3}$ is not differentiable at $x = 0$.



2 Relationship between Differentiability and Continuity

Theorem 2.1. *If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.*

3 Questions

1. Graph the function defined by

$$g(r) = \begin{cases} 1 + \cos(\pi r/2) & -2 \leqslant r \leqslant 2 \\ 0 & r < -2 \text{ or } r > 2 \end{cases}$$

(a) Is g continuous at $r = 2$? Explain your answer.

(b) Do you think g is differentiabe at $r = 2$? Explain your answer.

2. Decide whether each of the following functions is differentiable.

(a) $f(x) = \begin{cases} -2x & x < 0 \\ x^2 & x \geqslant 0 \end{cases}$

(b) $f(x) = (x + |x|)^2 + 1$

3. In each of the following cases, sketch the graph of a continuous function $f(x)$ with the given properies.

(a) $f''(x) > 0$ for $x < 2$ and for $x > 2$ and $f'(2)$ is undefinied.

(b) $f''(x) > 0$ for $x < 2$ and $f''(x) < 0$ for $x > 2$ and $f'(2)$ is undefined.

4. An electric charge Q in a circuit is given as a function of time t, by

$$Q = \begin{cases} C & t \leqslant 0 \\ Ce^{-t/RC} & t > 0 \end{cases}$$

where C and R are positive constants. The electric current I is the rate of change of charge, so $I = \frac{dQ}{dt}$. [Hint: To graph this function, take, for example, $C = 1$ and $R = 1$.]

(a) Is the charge Q a continuous function of time?

(b) Do you think the current I is defined for all times t?

5. A cable is made of an insulating material in the shape of a long, thin cylinder of radius r_0. It has electric charge distributed evenly throughout it. The electric field E at a distance r from the center of the cable is given by

$$E = \begin{cases} kr & r \leqslant r_0 \\ k\frac{r_0^2}{r} & r > r_0 \end{cases}$$

[Hint: To graph the function, take for example, $k = 1$ and $r_0 = 1$.]

(a) Is E continuous at $r = r_0$?

(b) Is E differentiable at $r = r_0$?

6. The potential φ of a charge distribution at a point on the y-axis is given by

$$\varphi = \begin{cases} 2\pi\sigma\left(\sqrt{y^2 + a^2} - y\right) & y \geqslant 0 \\ 2\pi\sigma\left(\sqrt{y^2 + a^2} + y\right) & y < 0 \end{cases}$$

where σ and a are positive constants [Hint: To graph the function, take for example, $2\pi\sigma = 1$ and $a = 1$.]

(a) Is φ continuous at $y = 0$?

(b) Is φ differentiable at $y = 0$?