The Second Derivative

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February 06, 2020

1 Higher Derivatives

1.1 Notation

Since the derivative itself is a function, we can consider the derivative of it. It is called the second derivative of the original function. Notationally, let the function be $y = f(x)$, then

$$f'(x) \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{means the first derivative,}$$

and

$$f''(x) \quad \text{or} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \text{means the second derivative.}$$

Similarily, by keeping taking derivatives, we can talk about all higher derivatives. The third derivative, which by definition is the derivative of the second derivative, is denoted $f'''(x)$ or $f^{(3)}(x)$ or $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$. And the nth derivative is

$$f^{(n)}(x) \quad \text{or} \quad \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$

2 The Second Derivative

2.1 Meaning

What does derivative tell us?

Recall that for the (first) derivative f' of f, we have

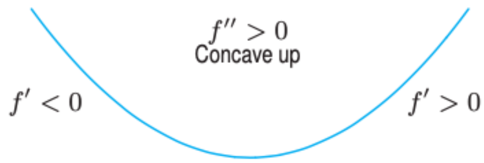
- If $f' > 0$ on an interval, then f is *increasing* over that interval.
- If $f' < 0$ on an interval, then f is *decreasing* over that interval.

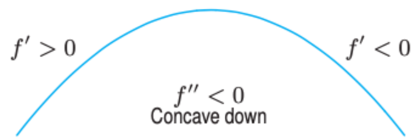
Then we should have

- If $f'' > 0$ on an interval, then f' is *increasing* over that interval.
- If $f'' < 0$ on an interval, then f' is *decreasing* over that interval.

But also notice that if f' is increasing, then f is concave _______________, and if f' is decreasing, f is concave _______________. Hence we have

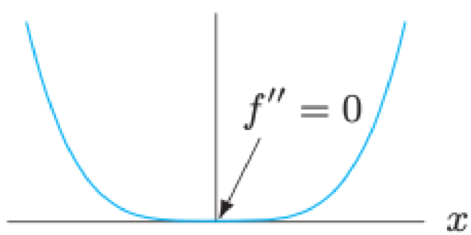
- If $f'' > 0$ on an interval, then f is ______________ over that interval.
- If $f'' < 0$ on an interval, then f is ______________ over that interval.





Conversely, we have

- If f is concave up and f'' exists on an interval, then f''______________0 there.
- If f is concave down and f'' exists on an interval, then f''______________0 there.



2.2 Velocity and Acceleration

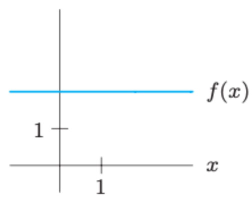
If $y = s(t)$ is the position function of an object at time t, then

- Velocity: $v(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = s'(t)$.
- Acceleration: $a(t) = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = s''(t) = v'(t)$

3 Questions

1. Give the sign of the first and secod derivatives for the function below.

8.



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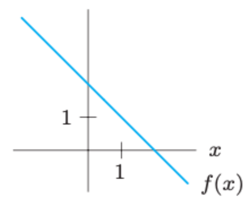

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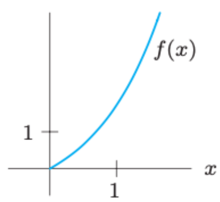

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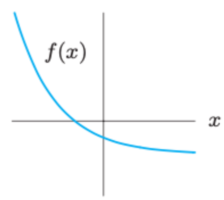

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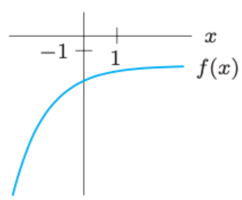

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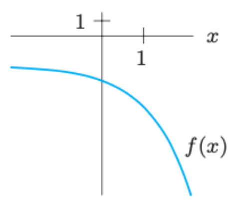


2. The table gives the number of passenger cars $C = f(t)$ in millions, in the US in the year t.

t	1975	1980	1985	1990	1995	2000	2005
C	106.7	121.6	127.9	133.7	128.4	133.6	136.6

   (a) Do $f'(t)$ and $f''(t)$ appear to be positive or negative during the period 1975-1990?

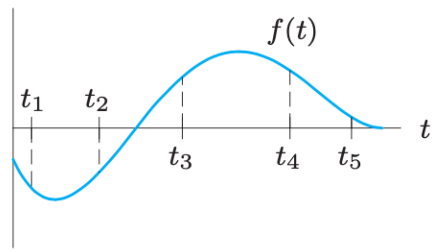
   (b) Do $f'(t)$ and $f''(t)$ appear to be positive or negative during the period 1990-2000?

   (c) Estimate $f'(2005)$. Using units, interpret your answer in terms of passenger cars.

3. Let $P(t)$ represent the price of a share of stock of a corporation at time t. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

   1. "The price of the stock is rising faster and faster."
   2. "The price of the stock is close to bottoming out."

4. The position $f(t)$ of a particle at time t is graphed below.

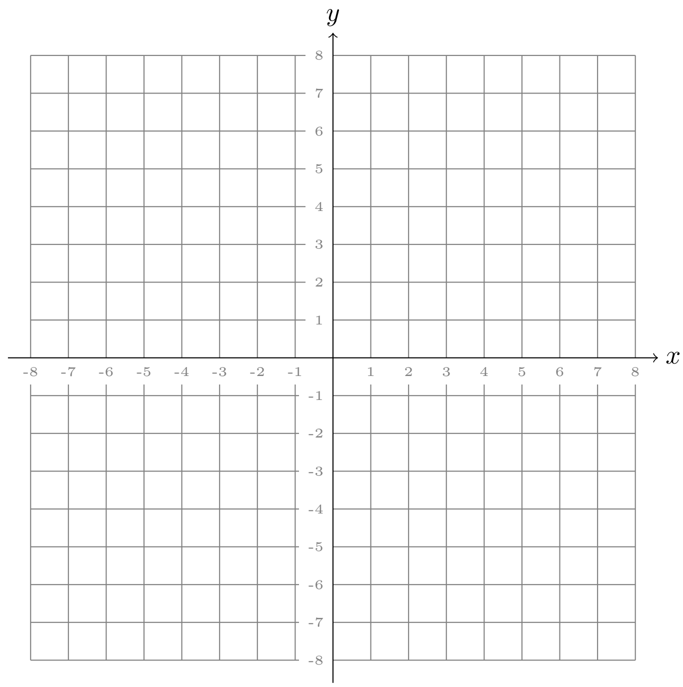


At which of the marked values of t can the following statements be true?

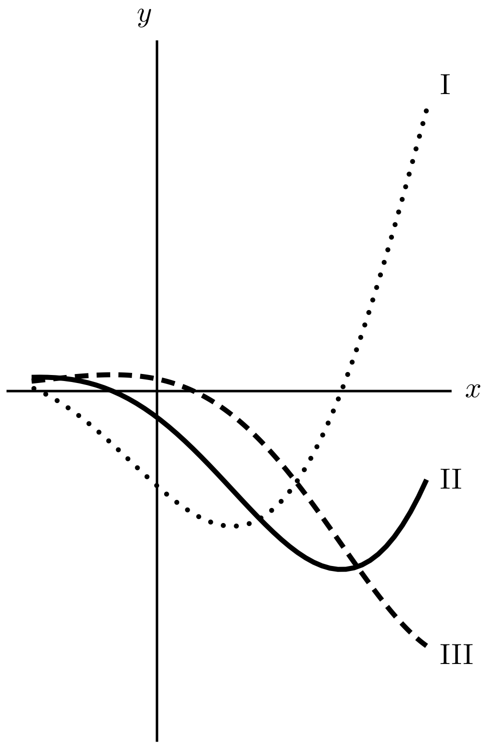
1. The position is positive.
2. The velocity is positive.
3. The acceleration is positive.
4. The position is decreasing.
5. The velocity is decreasing.

5. Sketch a *continuous* function f that has all the indicated properties.

- $f'(x) > 0$ for $-2 < x < 2$.
- $f'(x) < 0$ for $-6 < x < -2$ and $2 < x < 6$.
- $f''(x) > 0$ for $-3 < x < 0$ and $3 < x < 6$.
- $f''(x) < 0$ for $-6 < x < -3$ and $0 < x < 3$.
- $\lim_{x \to -\infty} f(x) = 2$ and $\lim_{x \to \infty} f(x) = 3$.


y
x
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7 8

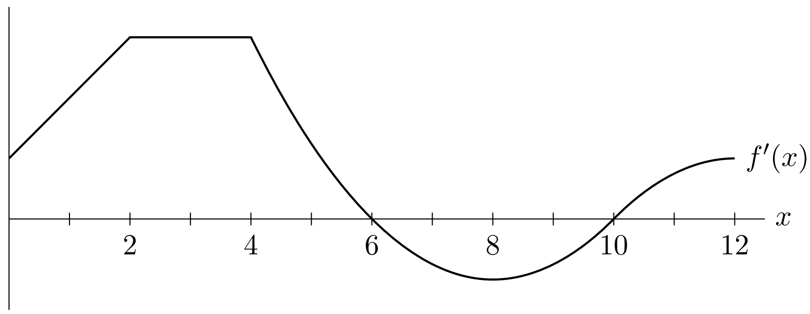
10. [5 points] Shown on the axes below are the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$.



Determine which graph is which and circle the ONE correct response below.

- $f(x)$: I, $f'(x)$: II, and $f''(x)$: III
- $f(x)$: I, $f'(x)$: III, and $f''(x)$: II
- $f(x)$: II, $f'(x)$: I, and $f''(x)$: III
- $f(x)$: II, $f'(x)$: III, and $f''(x)$: I
- $f(x)$: III, $f'(x)$: I, and $f''(x)$: II
- $f(x)$: III, $f'(x)$: II, and $f''(x)$: I

10. [10 points] The graph of $f'(x)$, the ***derivative*** of a function $f(x)$, is shown below.



For each of the following questions, circle ALL correct answers. You do not need to show work for this problem.

a. [2 points] On which of the following intervals is $f(x)$ increasing?

$0 < x < 2$ $2 < x < 4$ $4 < x < 6$ $6 < x < 8$ $8 < x < 10$ $10 < x < 12$

b. [2 points] On which of the following intervals is $f(x)$ concave down?

$0 < x < 2$ $2 < x < 4$ $4 < x < 6$ $6 < x < 8$ $8 < x < 10$ $10 < x < 12$

c. [2 points] On which of the following intervals is $f(x)$ linear?

$0 < x < 2$ $2 < x < 4$ $4 < x < 6$ $6 < x < 8$ $8 < x < 10$ $10 < x < 12$

d. [2 points] On which of the following intervals is $f''(x)$ increasing?

$0 < x < 2$ $2 < x < 4$ $4 < x < 6$ $6 < x < 8$ $8 < x < 10$ $10 < x < 12$

e. [2 points] Suppose $f(0) = -4$. Which of the following statements could be true?

$f(6) < -4$ $f(6) = -4$ $f(6) > -4$

9. [9 points] A pharmaceutical company just released a new medication to reduce the cold symptoms in children between 18 months old and 12 years of age. Let $D(z)$ be the dose (in ounces) recommended for a child that is z years old. A table with some values of $D(z)$ is shown below.

z	1.5	3	5	8	10	12
$D(z)$	2	5.2	8.6	11.4	14.5	20.2

a. [3 points] Find a formula for $D(z)$ on $3 \leq z \leq 5$ assuming it is a linear function in this interval.

Answer: $D(z) =$ ________________________

b. [3 points] Suppose that $D(z)$ is invertible. Give a practical interpretation of the equation

$$D^{-1}(9) = 6.5.$$

c. [3 points] Below is the first part of a sentence that will give a practical interpretation of the equation $D'(2) = 1.2$ in the context of this problem. Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include the appropriate units in your answer.

As the age of an child increases from 2 years to 25 months, the recommended dose ...