

Interpretations of the Derivative

Zhan Jiang

February 04, 2020

1 An alternative notation for the derivative

For function $y = f(x)$, following Leibniz, we wrote

$$\frac{dy}{dx} = f'(x)$$

This could be thought as the limit version of $\frac{\Delta y}{\Delta x}$.

We also use

$$\frac{d}{dx}(y) \quad \text{meaning "the derivative with respect to } x \text{ of } y$$

Finally, in order to express $f'(2)$ using Leibniz's notation, we use

$$\left. \frac{dy}{dx} \right|_{x=2}$$

2 Interpretation of derivatives

An example of interpreting derivatives.

A cup of hot coffee is left on a table. The temperature of the coffee C in degrees Fahrenheit, is given by $C = f(t)$, where t is the time in minutes since the coffee was put on the table. Give a practical interpretation of $f'(5) = -2$.

- (a) After 5 minutes, the temperature of the coffee is changing by about $\frac{1}{3}^\circ$ F in the next ten seconds.
- (b) The temperature of the coffee is decreasing by 2° F every minute at $t = 5$ minutes.
- (c) During its fifth minute on the table, the temperature of the coffee decrease by approximately 2° F.
- (d) The temperature of the coffee is decreasing by 2° F per minute five minutes after it was put on the table.
- (e) After the coffee has been on the table for 5 minutes, its temperature decrease by 2 in the next minute.

Key features:

- A complete, comprehensive English sentence that could be understood by someone who knows no calculus.
- States the given input value and indicating a small change in the independent variable from that input value.
- Appropriate use of numbers and units.
- Indicates appropriate direction of change in the output (increase or decrease).
- Indicates that the change in output is approximate.

QUESTIONS:

1. The time for a chemical reaction, T (in minutes), is a function of the amount of catalyst present, a (in milliliters), so $T = f(a)$. If $f'(5) = -3$, what are the units of 5? What are the units of -3?

Interpretation of $f'(5) = -3$: When there are 5 milliliters of catalyst present, if we increase the catalyst by 0.1 milliliters, then the time of the chemical reaction will _____ by approximately _____ seconds.

2. The cost C (in dollars) to produce q quarts of ice cream is $C = f(q)$.

Interpretation of $f'(200) = 2$: When producing 200 quarts of ice cream, it will cost approximately _____ dollar(s) more to produce one extra quart of ice cream.

3. The population of Mexico in millions is $P = f(t)$, where t is the number of years since 1980. Explain the meaning of the statements:

(a) $f'(6) = 2$.

(b) $f^{-1}(95.5) = 16$

(c) $(f^{-1})'(95.5) = 0.46$

5. (12 points) At woot.com the staff has become quite good at predicting the number of items that will be sold based on the brand name, reliability reports, the price, and the predicted popularity of the item. The maximum number of items, N , that they expect to sell during the entire sale period on a given day is a function of what they call the Max Sales Index, i , so $N = f(i)$, where the units of i are referred to as “points.”

(a) In the context of this problem, give a practical interpretation of $f(10)$.

(b) In the context of this problem, what is the practical interpretation of $f'(5) = 2500$?

(c) The number of Wooters (registered members of Woot.com) is currently over 500,000. Since there is not a mechanism for “un-registering,” and the membership has grown very quickly, assume that the number of Wooters, W in thousands, is an invertible function of time, t , in hours, $W = g(t)$. In this context, give a practical interpretation of $(g^{-1})'(200) = .05$

8. (12 points) The potential energy E , in joules, of an object above the Earth's surface is a function of the distance, h , in meters, of the object from the surface of the Earth. That is, $E = f(h)$.

(a) In the context of this problem, explain the meaning of $f(20) = 1000$?

(b) In the context of this problem, explain the meaning of $f'(9) = 50$?

(c) In the context of this problem, explain the meaning of $f^{-1}(150) = 3$?

(d) In the context of this problem, explain the meaning of $(f^{-1})'(400) = \frac{1}{50}$?

7. (11 points) Suppose $P = m(t)$ is the population of Mexico in millions, where t is the number of years since 1980. Explain the meaning of the statements below. You should only use terms and phrases understandable to someone who has never taken calculus. (Assume that the population function is invertible.)

(a) (3 pts.) $\left. \frac{dP}{dt} \right|_{t=0} > 0$

(b) (4 pts.) $m^{-1}(97.5) = 18$

(c) (4 pts.) $(m^{-1})'(97.5) = 0.46$

2. [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges p cents per bottle. You may assume $b(p)$ is differentiable and invertible.
- a. [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.
- b. [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.
- c. [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.
- d. [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of d dollars per bottle? (Circle your answer.)

$$h(d) = 100b(d) \quad h(d) = \frac{b(d)}{100} \quad h(d) = b(100d) \quad h(d) = b\left(\frac{d}{100}\right)$$

5. [11 points] Oren, a Math 115 student, realizes that the more caffeine he consumes, the faster he completes his online homework assignments. Before starting tonight's assignment, he buys a cup of coffee containing a total of 100 milligrams of caffeine.

Let $T(c)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes c milligrams of caffeine. Suppose that T is continuous and differentiable.

- a. [2 points] Circle the ONE sentence below that is best supported by the statement "the more caffeine he consumes, the faster he completes his online homework assignments."
- i. $T'(c) \geq 0$ for every value c in the domain of T .
 - ii. $T'(c) \leq 0$ for every value c in the domain of T .
 - iii. $T'(c) = 0$ for every value c in the domain of T .
- b. [1 point] Explain, in the context of this problem, why it is reasonable to assume that $T(c)$ is invertible.

- c. [2 points] Interpret the equation $T^{-1}(100) = 45$ in the context of this problem. Use a complete sentence and include units.

- d. [3 points] Suppose that p and k are constants. In the equation $T'(p) = k$, what are the units on p and k ?

Answer: Units on p are _____

Answer: Units on k are _____

- e. [3 points] Which of the statements below is best supported by the equation $(T^{-1})'(20) = -10$? Circle the ONE best answer.
- i. If Oren has consumed 20 milligrams of caffeine, then consuming an additional milligram of caffeine will save him about 10 minutes on tonight's assignment.
 - ii. The amount of caffeine that will result in Oren finishing his homework in 21 minutes is approximately 10 milligrams greater than the amount of caffeine that Oren will need in order to finish his homework in 20 minutes.
 - iii. The rate at which Oren is consuming caffeine 20 minutes into his homework assignment is decreasing by 10 milligrams per minute.
 - iv. In order to complete tonight's assignment in 19 rather than 20 minutes, Oren needs to consume about 10 milligrams of additional caffeine.
 - v. If Oren consumes 20 milligrams of caffeine, then he will finish tonight's assignment approximately 10 minutes faster than if he consumes no caffeine.