The Derivative Function

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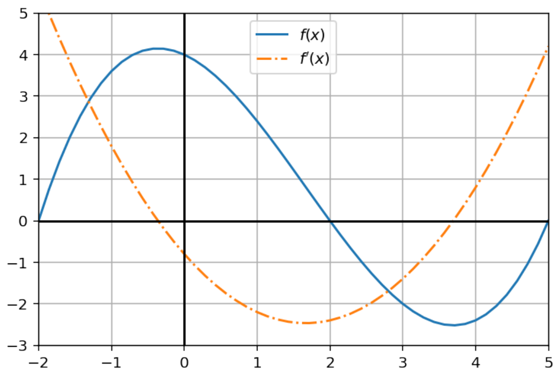
1 Definition

For any function f, we define the *derivative function* f' by

$$f'(x) = \text{rate of change of } f \text{ at } x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2 Graphically



If $f' > 0$ on an interval, then f is *increasing* over that interval.

If $f' < 0$ on an interval, then f is *decreasing* over that interval.

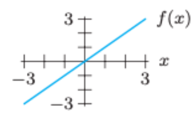
1. In each case, graph a smooth curve whose slope meets the condition.
   (a) Everywhere positive and increasing gradually.
   (b) Everywhere positive and decreasing gradually.
   (c) Everywhere negative and increasing gradually (becoming less negative).
   (d) Everywhere negative and decreasing gradually (becoming more negative).

2. Draw a possible graph of $y = f(x)$ given the following information about its derivative.

- $f'(x) > 0$ for $x < -1$.
- $f'(x) < 0$ for $x > -1$.
- $f'(x) = 0$ for $x = -1$.

3. Sketch the graph of $f'(x)$ given the graph of $f(x)$ for each graph below.

29.



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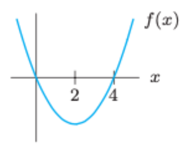

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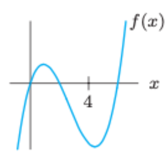

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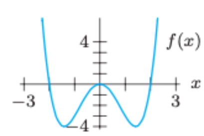

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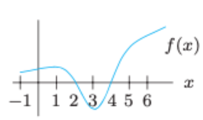


3 Formula

It is immediate from the definition that

- If $f(x) = k$, then $f'(x) = 0$.
- If $f(x) = mx + b$, then $f'(x) = m$.

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. How to prove this?

Write $f_n(x) = x^n$, then

$$\begin{aligned}
f_n(x)' &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+h)^{n-1} - x^n}{h} \\
&= \lim_{h\to 0} \frac{x(x+h)^{n-1} + h(x+h)^{n-1} - x^n}{h} \\
&= \lim_{h\to 0} \frac{x(x+h)^{n-1} - x^n}{h} + \lim_{h\to 0}(x+h)^{n-1} \\
&= x \lim_{h\to 0} \frac{(x+h)^{n-1} - x^{n-1}}{h} + x^{n-1} \\
&= x f'_{n-1}(x) + x^{n-1}
\end{aligned}$$

If we already know that $f'_{n-1}(x) = (n-1)x^{n-2}$. Then

$$f'_n(x) = x * (n-1)x^{n-2} + x^{n-1} = (n-1)x^{n-1} + x^{n-1} = nx^{n-1}$$

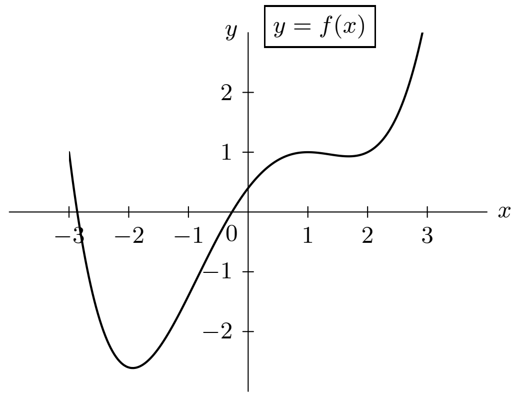
Another apporach is using the binomial expansion formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = a^n + na^{n-1}b + \cdots + b^n$$

So

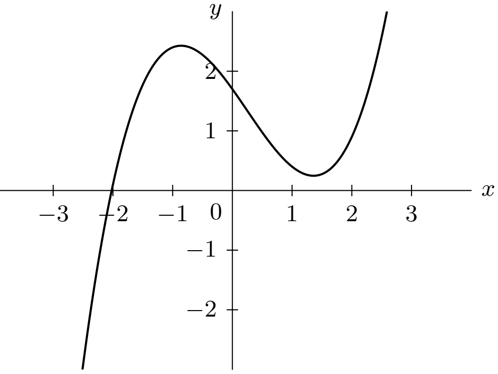
$$\begin{aligned}
f_n(x)' &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \cdots + h^n - \cancel{x^n}}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \cdots + h^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + \cdots + h^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

1. [5 points] Below is the graph of a function $f(x)$.

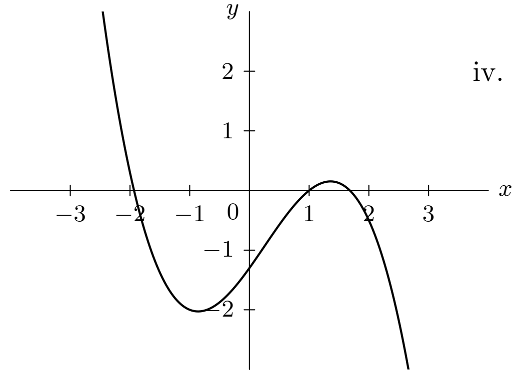


There are six graphs shown below. Circle the <u>one</u> graph that could be the graph of the derivative $f'(x)$.

i.



iv.



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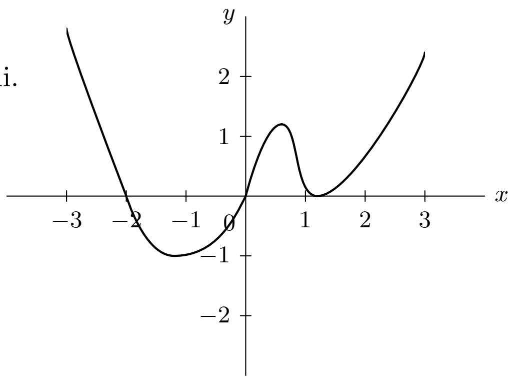
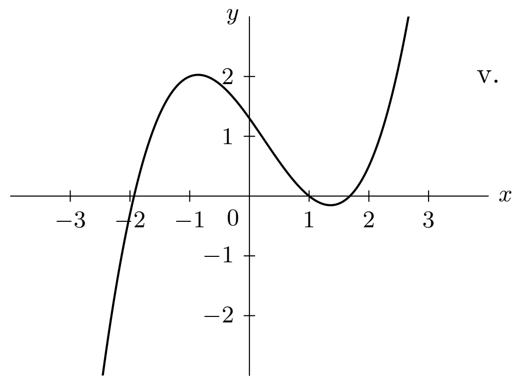
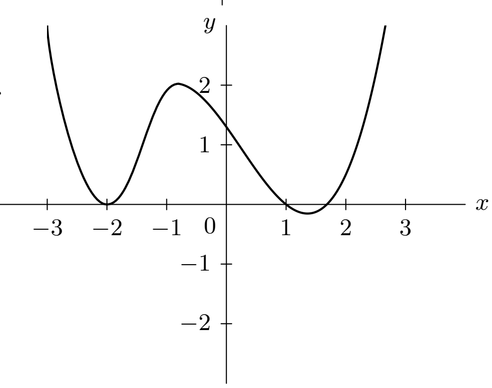


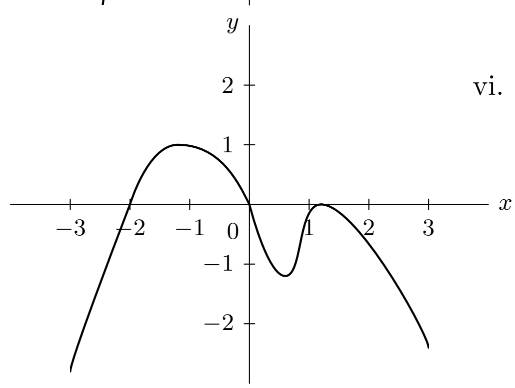
v.



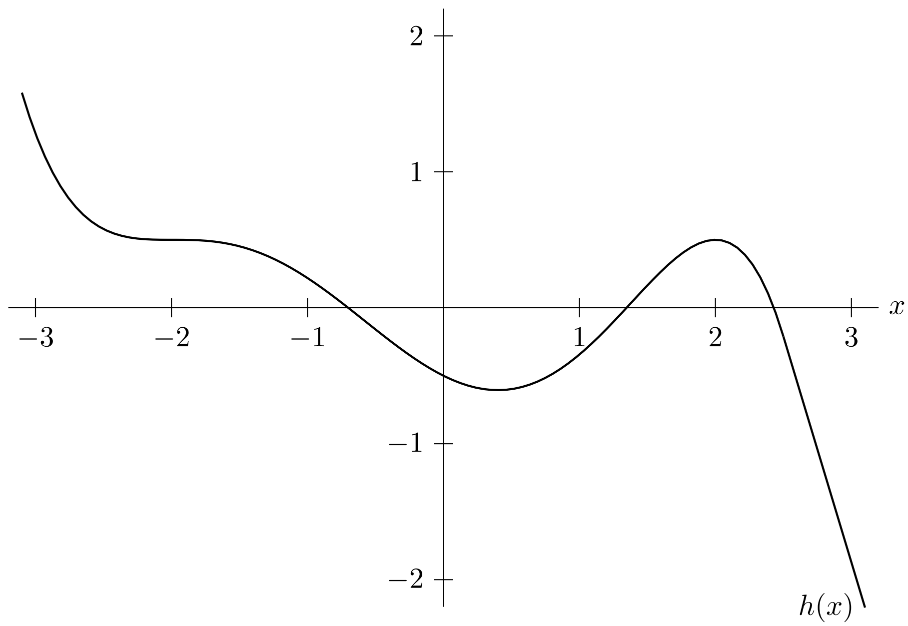
iii.

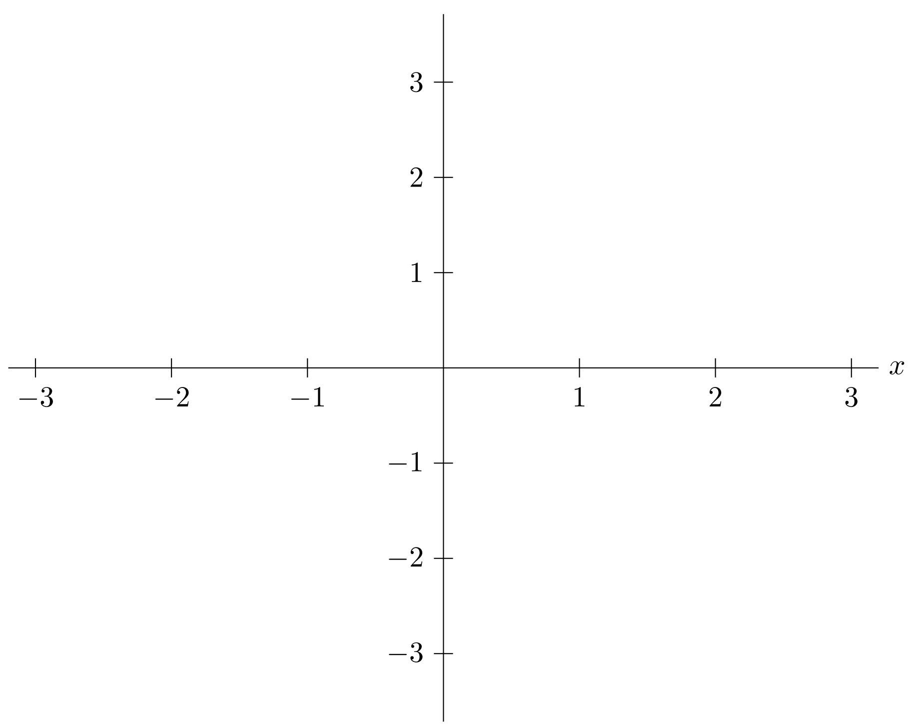


vi.



9. [10 points] Given below is the graph of a differentiable function $h(x)$ which is linear for $x > 2.5$. On the second set of axes, sketch a possible graph of $h'(x)$. Be sure your graph is drawn carefully.

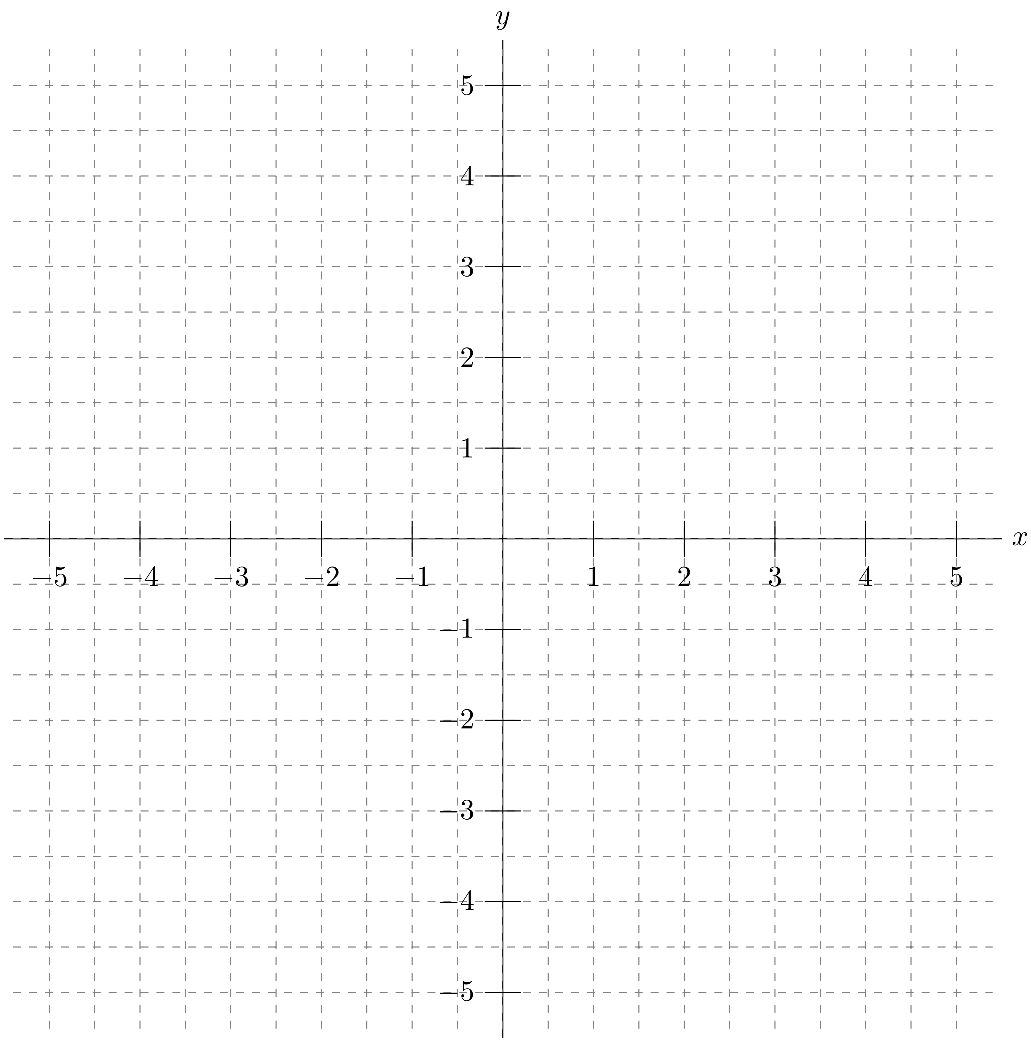




6. [12 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all of the following:

- $h(x)$ is defined for all x in the interval $-5 < x < 5$.
- $h'(x) > 0$ for all $x < -3$.
- $\lim_{x \to -2} h(x) = 0$.
- $h(-2) = -3$.
- The average rate of change of $h(x)$ between $x = -1$ and $x = 1$ is 2.
- $h(1) = 2$.
- $h(x)$ is linear between $x = 1$ and $x = 3$.
- $h'(2) = -1$.
- $\lim_{x \to 4^-} h(x) = -1$.
- $\lim_{x \to 4} h(x)$ does not exist.
- $h'(x) < 0$ for all $x > 4$.

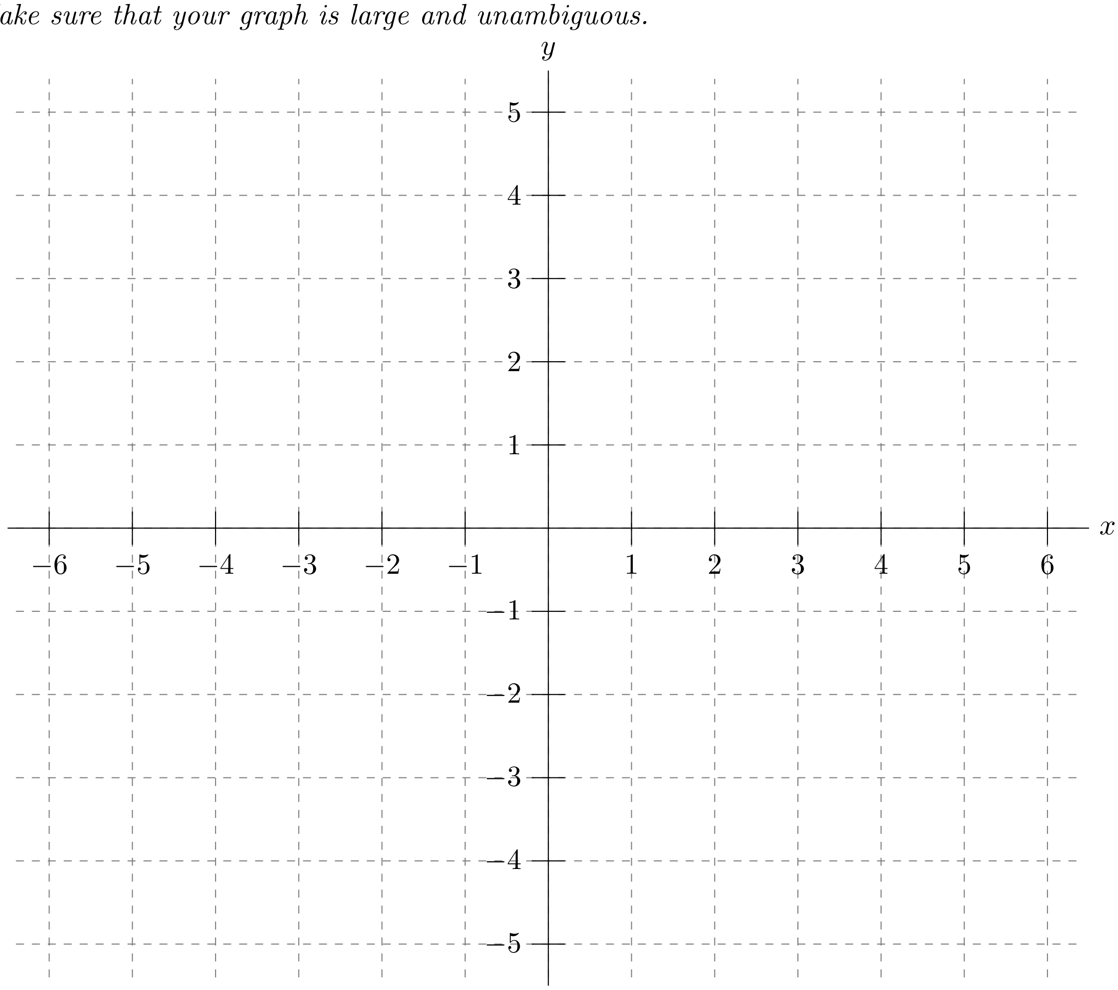
Make sure that your sketch is large and unambiguous.



6. [12 points] On the axes provided below, sketch the graph of a single function $y = g(x)$ satisfying all of the following:

- $g(x)$ is defined for all x in the interval $-6 < x < 6$.
- For all x in the interval $-6 < x < -4$, the function $g(x)$ is continuous at x and $g'(x) > 0$.
- $g(-4) = -1$.
- $\lim_{x \to -4^+} g(x) = 2$.
- $g(-3) = 1$.
- $g(-2) = -1$.
- The function $g(x)$ is continuous on the interval $[-3, -1]$.
- The average rate of change of $g(x)$ between $x = -3$ and $x = -1$ is 2.
- $g'(1) = 0$.
- $g(x)$ is <u>not</u> continuous at $x = 2$.
- The function $g(x)$ is continuous on the interval $3 < x < 6$.
- The slope of the tangent line to the graph of $y = g(x)$ at $x = 3$ is positive.
- $g(x)$ is increasing and concave down on the interval $4 < x < 6$.

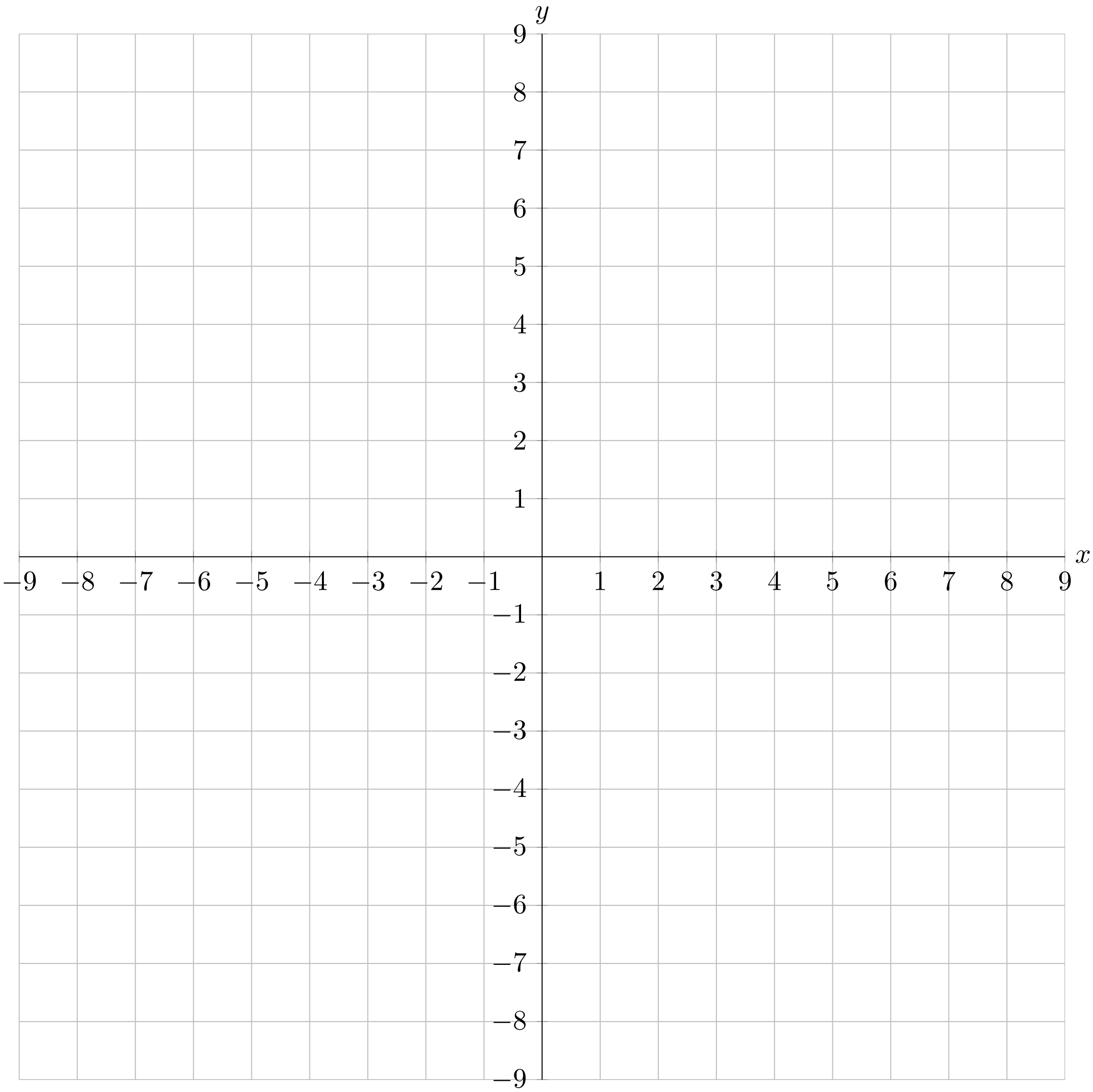
Make sure that your graph is large and unambiguous.



8. [11 points] On the axes provided below, sketch the graph of a single function $y = f(x)$ satisfying all of the following conditions:

- The domain of $f(x)$ is the interval $-8 < x \leq 6$.
- $f(x)$ is continuous for all x in the interval $-8 < x < -2$.
- $f'(-7) = 0$.
- $f(x)$ is decreasing and concave up for all x in the interval $-6 < x < -4$.
- The average rate of change of $f(x)$ is equal to 0.5 between $x = -5$ and $x = -2$.
- $f(0) = 2$ and $f'(0) = -1$.
- $\lim_{x \to 2^-} f(x) = f(2)$ and $\lim_{x \to 2^+} f(x) < \lim_{x \to 2^-} f(x)$.
- $f(x)$ has constant rate of change on the interval $3 \leq x \leq 6$.

Make sure that your graph is large and unambiguous.



10. [10 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all of the following:

- $h(x)$ is defined for all x in the interval $-6 < x < 6$.
- $h'(x) < 0$ for all $x < -3$.
- $\lim_{x \to -2^+} h(x) = -1$.
- $h'(0) = 0$.
- The average rate of change of $h(x)$ between $x = -1$ and $x = 2$ is 1.
- $h(x)$ is not continuous at $x = 3$.
- $h(x) > 0$ for all $x > 3$.
- $h'(x) > 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.