The Derivative at a Point

Zhan Jiang

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1 The derivative at a point

1.1 Definition

Recall that for a function $f(x)$,we have "average velocity" formula

$$\frac{f(b) - f(a)}{b - a}.$$

If the function $f(x)$ is no longer a position function, we use this formula to mean the *average rate of change*.

If the interval is $(a, a + h)$ (or $(a + h, a)$ if h is negative), then we have

$$\frac{f(a + h) - f(a)}{h}.$$

If we let $h \to 0$, this is the "instantaneous velocity", or *instantaneous rate of change* of $f(x)$. Iħis instantaneous rate of change is called the *derivative* of $f(x)$ at $x = a$.

To summarize, we have following definition

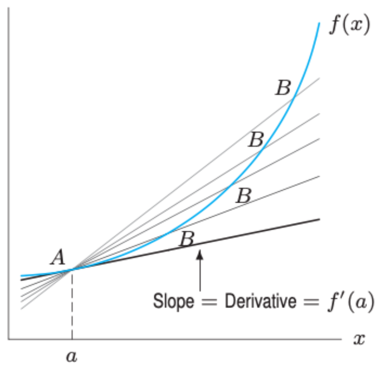
Definition 1.1. The *derivative of* f *at* a, written $f'(a)$, is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

If the limit exists, then f is said to be *differentiable at* a.

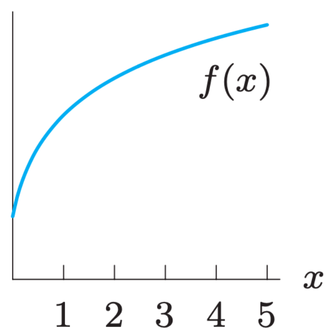
1.2 Visualizing

The derivative $f'(a)$ is the slope of the tangent line at $x = a$ of the graph $y = f(x)$.



$f'(a)$ = the slope of the curve at $(a, f(a))$ = the slope of the tangent line to the curve at $(a, f(a))$.

1. Show how to represent the following on figure provided below



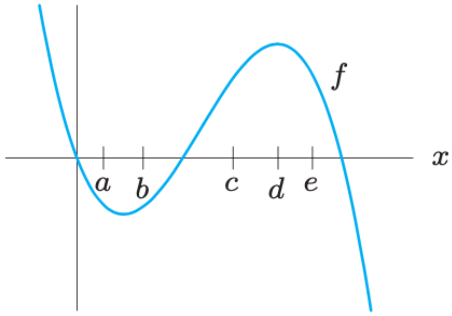
(a) $f(4)$

(b) $f(4) - f(2)$

(c) $\dfrac{f(5) - f(2)}{5 - 2}$

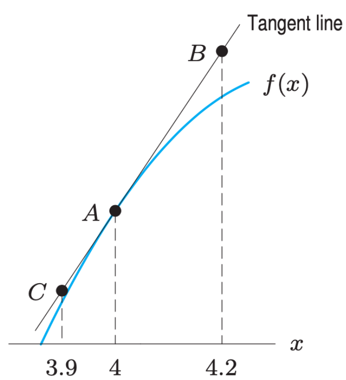
(d) $f'(3)$

2. Below is a graph of $f(x)$. Match the derivatives in the table with the points a, b, c, d, e.



x	$f'(x)$
	0
	0.5
	2
	−0.5
	−2

3. The function below has $f(4) = 25$ and $f'(4) = 1.5$. Find the coordinates of the points A, B, C.



2 Computing the derivative algebraically

Take $f(x) = \frac{1}{x}$ at $x = 1$ as an example:

$$\begin{aligned} f'(x)|_{x=1} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{1/(1+h) - 1}{h} \\ &= \lim_{h \to 0} \frac{1 - (1+h)}{h(1+h)} \\ &= \lim_{h \to 0} \frac{-\not{h}}{\not{h}(1+h)} \\ &= \lim_{h \to 0} \frac{-1}{1+h} \\ &= -1 \end{aligned}$$

1. Find the derivative $f'(1)$ for following functions
   (a) $f(x) = \dfrac{1}{x+1}$
   (b) $f(x) = \dfrac{1}{x^2}$
   (c) $f(x) = \sqrt{x}$

4. [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

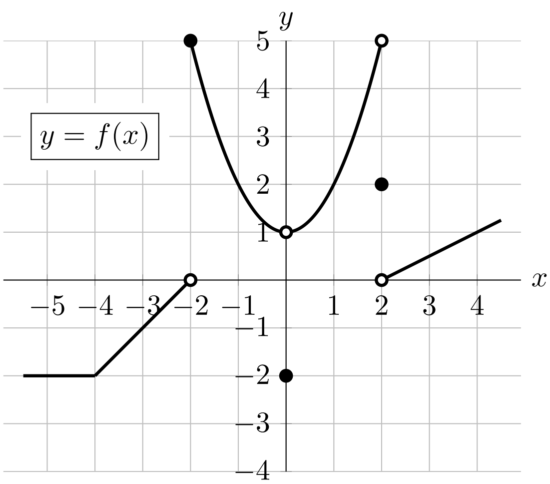
a. [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter P.
Do not attempt to evaluate or simplify the limit.

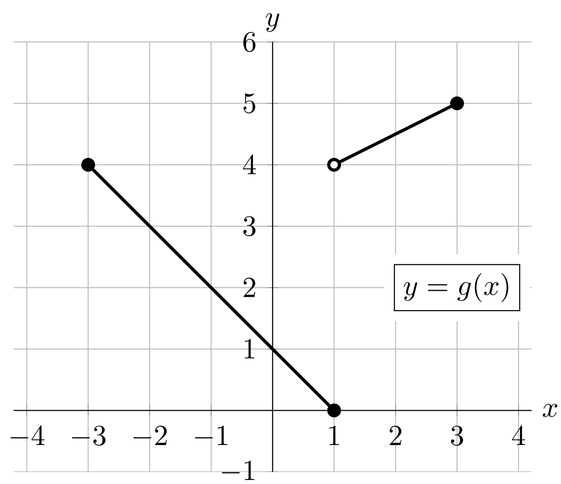
$P'(0) =$ ______________________________

b. [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
Be sure to include enough clear graphical or numerical evidence to justify your answer.

Answer: $P'(0) \approx$ ______________________________

The graphs of the functions $f(x)$ and $g(x)$ are included here for your convenience.





g. [3 points] Find all the values of x with $-5 < x < 4$ at which the function $f(x)$ is <u>not</u> continuous.

Answer: ____________________

h. [2 points] What is the range of $y = g(x)$?

Answer: ____________________

i. [2 points] For which of the following values of x is $f'(x) > 0$? Circle all that apply.

$x = -5$ $\qquad$ $x = -1$ $\qquad$ $x = 1.5$ $\qquad$ $x = e$ $\qquad$ NONE OF THESE

2. [5 points] Let

$$K(p) = (1 + \cos(p))^{1+2p}.$$

Use the limit definition of the derivative to write an explicit expression for $K'(4)$. *Your answer should not involve the letter K. Do not attempt to evaluate or simplify the limit. Please write your final answer in the answer box provided below.*

Answer: $K'(4) =$

b. [5 points] After climbing the globe, Gabe jumps onto a small ferris wheel. Let $H(t)$ be his height, in inches, above the ground t seconds after Gabe jumped, where

$$H(t) = 12 + 9\cos\left(\frac{\pi}{75}(t-120)\right).$$

Find the the *smallest* positive value of t at which Gabe's height above the ground is 10.5 inches. Clearly show each step of your algebraic work. Give your answer in *exact* form.

Answer: $t =$ ________________________

3. [5 points] Let

$$B(k) = e^{-4k^2}\tan(k+3).$$

Use the limit definition of the derivative to write an explicit expression for $B'(5)$. *Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Answer: $B'(5) =$

7. [9 points] For a constant c, let

$$K(x) = \frac{2^{cx}}{e^{x-c}}.$$

a. [5 points] Use the limit definition of the derivative to write an explicit expression for $K'(3)$. *Your answer may include the constant c but should not involve the letter K. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $K'(3) =$

b. [4 points] Find the value of c so that $K(1) = 5$. Give your answer in **exact form** and show all your work.

Answer: $c =$ ______________________