Limits

Zhan Jiang

January 24, 2020

1 Limits

1.1 One-sided Limits

When we write $\lim_{x \to 2} f(x)$, we mean the number that $f(x)$ approaches as x approaches 2 *from both sides*.

For example, we have to examine values of x larger than 2, such as 2.1, 2.01, 2.0009, ... and values smaller than 2, such as 1.9, 1.99, 1.998, ...

If we only want to approach 2 only through values greater than 2, then we write

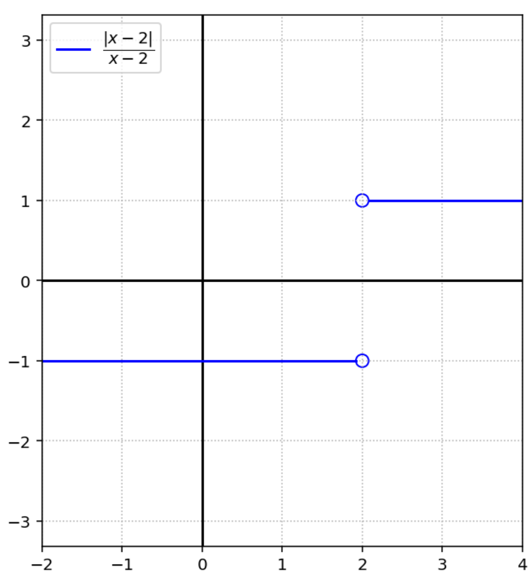
$$\lim_{x \to 2^+} f(x)$$

for the number that $f(x)$ approaches.

Similarly, if we only approach 2 through value smaller than 2, then we write

$$\lim_{x \to 2^-} f(x).$$

We call $\lim_{x \to 2^+} f(x)$ a *right-hand limit* and $\lim_{x \to 2^-} f(x)$ a *left-hand limit*.



1.2 Horizontal asymptotes and limits at ∞

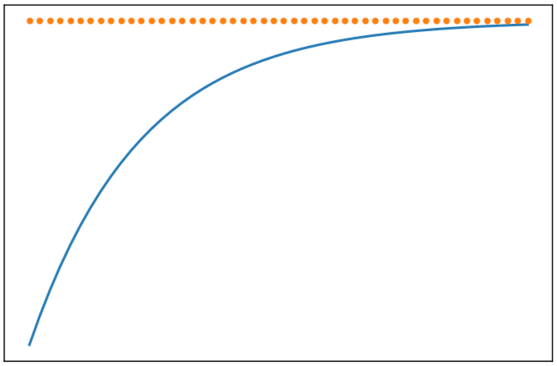
If $f(x)$ stays close to a number L when x is sufficiently large, then we write

$$\lim_{x \to \infty} f(x) = L.$$

Similarly, if $f(x)$ stays close to L when x is negative and has a sufficiently large absolute value, then we write

$$\lim_{x \to -\infty} f(x) = L.$$

If the limit of $f(x)$ as $x \to \infty$ or $x \to -\infty$ is L, we say that the graph of f has $y = L$ as a *horizontal asymptote*.



1.3 Vertical asymptotes and limits

If $x = a$ is a vertical asymptote of the graph, then we have

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

Again, $\lim f(x) = \pm\infty$ means the limit *does not exist.* It's only a notion to describe the behavior of $f(x)$.

2 Properties of Limits and Continuity

2.1 Limits

Theorem 2.1. *Assuming all the limits on the* right-hand side *exist:*

(1) If b is a constant, then $\lim_{x \to c}(bf(x)) = b\left(\lim_{x \to c} f(x)\right)$.

(2) $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

(3) $\lim_{x \to c}(f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$.

(4) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, *provided* $\lim_{x \to c} g(x) \neq 0$.

All the properties here hold for both one- and two-sided limits, as well as limits at infinity.

2.2 Continuity

Theorem 2.2. *If f and g are both continuous functions on an interval and b is a constant, then, on that same interval,*

(1) $bf(x)$ is continuous.

(2) $f(x) + g(x)$ is continuous.

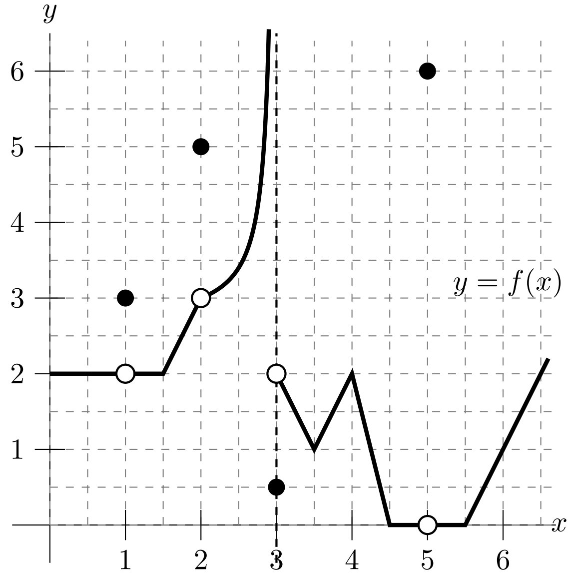
(3) $f(x)g(x)$ is continuous.

(4) $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

Theorem 2.3. *If f and g are continuous, then*

1. *The composition function $g(f(x))$ is continuous as long as it is defined.*
2. *If f has an inverse function f^{-1}, then f^{-1} is continuous.*

1. [10 points] A portion of the graph of a function f is shown below.



Note: You may assume that pieces of the function that appear linear are indeed linear.

Use the graph above to evaluate each of the expressions below, and write your answer on the answer blank provided. If any of the quantities do not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE.

a. [1 point] $f(1)$

Answer: ______________

b. [1 point] $\lim_{x\to 5} f(x)$

Answer: ______________

c. [1 point] $\lim_{q\to 3} f(q)$

Answer: ______________

d. [1 point] $\lim_{z\to 2} f(2)$

Answer: ______________

e. [1 point] $\lim_{r\to 6^-} f(r)$

Answer: ______________

h. [1 point] $\lim_{t\to 3} f(t)f(t+2)$

Answer: ______________

i. [1 point] $\lim_{x\to 3^+} f(f(x))$

Answer: ______________

j. [1 point] $\lim_{s\to 1} f(f(s))$

Answer: ______________

3. [9 points] Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{60(x^2 - x)}{(x^2+1)(3-x)} & \text{for } x < 2 \\ c & \text{for } x = 2 \\ 5e^{ax} - 1 & \text{for } x > 2 \end{cases}$$

where a and c are constants.

a. [5 points] Find values of a and c so that both of the following conditions hold.

- $\lim\limits_{x \to 2} h(x)$ exists.
- $h(x)$ is not continuous at $x = 2$.

Note that this problem may have more than one correct answer. You only need to find one value of a and one value of c so that both conditions above hold. Remember to show your work clearly.

Answer: $a =$ ________________ and $c =$ ________________

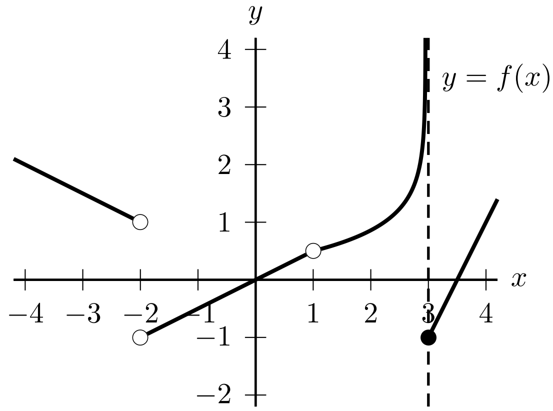
b. [2 points] Determine $\lim\limits_{x \to -\infty} h(x)$. If the limit does not exist, write DNE.

Answer: $\lim\limits_{x \to -\infty} h(x) =$ ________________

c. [2 points] Find all vertical asymptotes of the graph of $h(x)$. If there are none, write NONE.

Answer: Vertical asymptote(s): ________________

6. [11 points] Below is the graph of a portion of a function $f(x)$.



a. [2 points] Give all values of a in the interval $-4 < a < 4$ that are not in the domain of $f(x)$. If there are none, write NONE.

Answer: ______________________

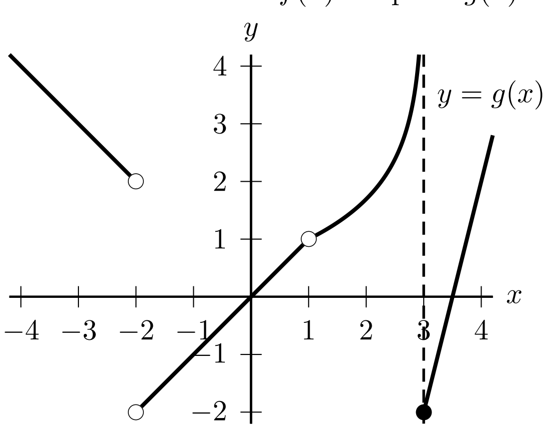
b. [2 points] Give all values of a in the interval $-4 < a < 4$ where $f(x)$ is not continuous at $x = a$. If there are none, write NONE.

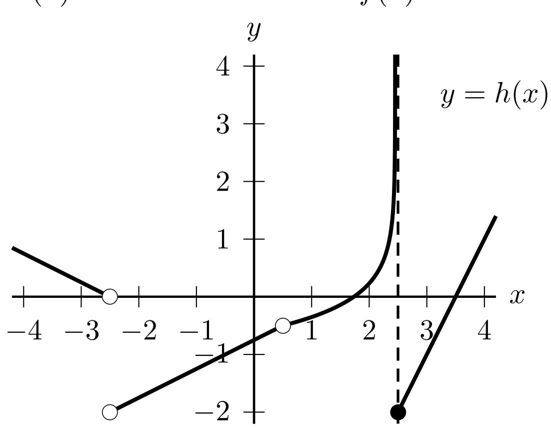
Answer: ______________________

c. [2 points] Give all values of a in the interval $-4 < a < 4$ where $\lim_{x \to a} f(x)$ does not exist. If there are none, write NONE.

Answer: ______________________

d. [5 points] The graphs below show portions of two other functions $g(x)$ and $h(x)$ which are transformations of $f(x)$. Express $g(x)$ and $h(x)$ as transformations of $f(x)$.

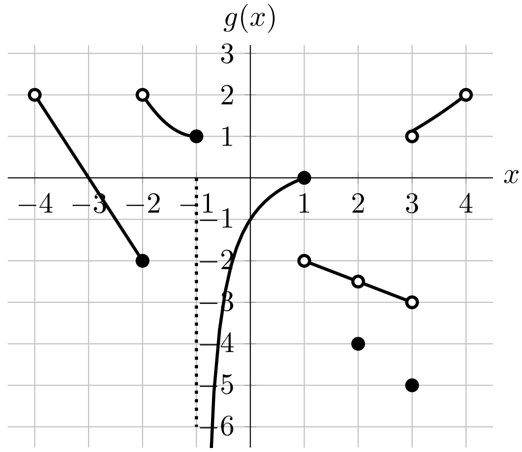




Answer: $g(x) =$ ______________ and $h(x) =$ ______________

8. [15 points] Consider the functions $f(x)$ and $g(x)$ given by the formula and graph below.

$$f(x) = \begin{cases} 2x^3 - 2x^2 & \text{for } x \le 1, \\ x^3 + 1 & \text{for } x > 1. \end{cases}$$



a. [5 points] Circle the correct answer(s) in each of the following questions.

i) At which of the following values of x is the function $g(x)$ not continuous?

$x = -3$ $\qquad$ $x = -1$ $\qquad$ $x = 0$ $\qquad$ $x = 2$ $\qquad$ $x = 3.5$

ii) At which of the following values of x is the function $f(x) + g(x)$ continuous?

$x = -2$ $\qquad$ $x = -1$ $\qquad$ $x = 0$ $\qquad$ $x = 1$ $\qquad$ $x = 2$

Note that $g(x)$ is linear on the interval $(-4, -2)$, $(1, 2)$ and $(2, 3)$. All your answers below should be *exact*. If any of the quantities do not exist, write DNE.

b. [2 points] Find $\lim_{x \to 2} (2f(x) + g(x))$.

Answer: ______________

c. [2 points] Find $\lim_{x \to \infty} \dfrac{f(2x)}{x^3}$.

Answer: ______________

d. [2 points] Find $\lim_{x \to \infty} g\left(x^2 e^{-x} + 3\right)$.

Answer: ______________

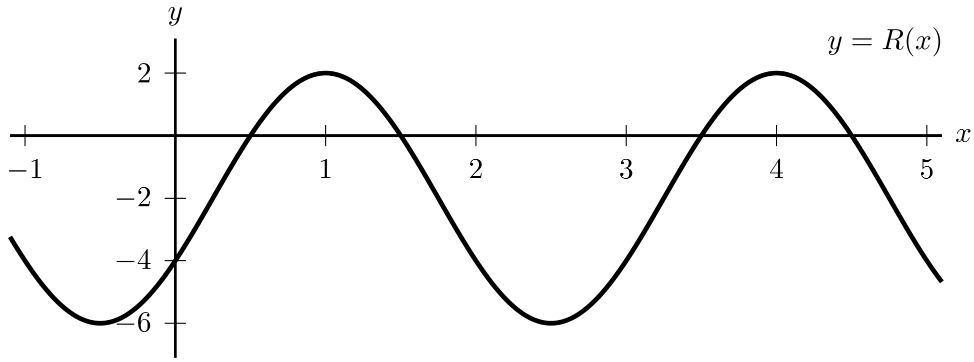
e. [2 points] For which value(s) of p does $\lim_{x \to p^+} g(x) = 1$?

Answer: ______________

f. [2 points] Find $\lim_{x \to -1^-} f(-x)$.

Answer: ______________

8. [6 points] Given below is the graph of a sinusoidal function $R(x)$.



Find a possible formula for $R(x)$.

Answer: $R(x) =$ ______________________________

9. [4 points] The table below gives several values of a function $w(x)$.

x	4.5	4.9	4.99	5	5.01	5.1	5.5
$w(x)$	-0.879	-0.154	-0.015	0	0.060	0.630	3.750

Use the information in the table above to estimate the following limit.

$$\lim_{h\to 0^-} \frac{w(5+h)}{h}$$

Clearly show any computations that you use to make this estimate.

Answer: $\displaystyle\lim_{h\to 0^-} \frac{w(5+h)}{h} \approx$ ______________________________