Limits and Continuity

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1 Limits

We write $\lim_{x \to c} f(x) = L$ if the value of $f(x)$ approach L as x approaches c. Intuitively, the value of $f(x)$ is as close to L as we want whenever x is sufficiently close to c.

1.1 Numerically

If we compute the value of $f(x)$ for a sequence of x really close to c, we should see a sequence of value really close to L.

Examples:

1. $f(x) = \dfrac{\sin(x)}{x}$ when $x \to 0$

x	0.1	0.01	0.001
$f(x)$	0.99833	0.99998	0.99999

2. $f(x) = \frac{e^x - 1}{x}$ when $x \to 0$

x	0.1	0.01	0.001
$f(x)$	1.0517	1.0050	1.0005

3. $f(x) = \dfrac{\cos(x)}{\sin(x)}$ when $x \to 2\pi \approx 6.2831853..$

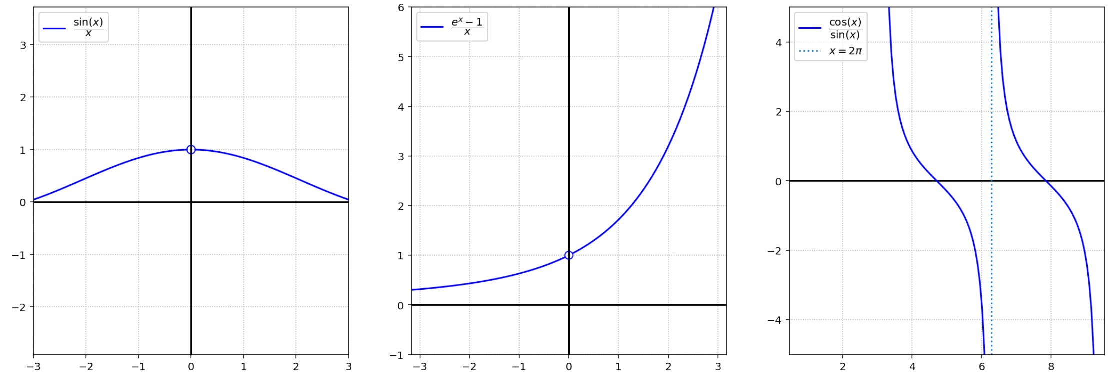
x	6.183	6.273	6.282
$f(x)$			

What is your estimation on $\lim_{x \to 0} f(x)$ based on following table.

x	-0.1	-0.01	0.01	0.1
$f(x)$	1.987	1.999	1.999	1.987

1.2 Graphically

We can also figure out the limit using a graph, see graphs for each function we discussed above.



2 Continuity

Definition 2.1. The function f is

- *continuous* at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x \to c} f(x) = f(c)$$

- continuous on an interval $[a, b]$ if it is continuous at every point in the interval.

So far, all functions we've learned, e.g. linear functions, exponential functions, logarithm functions, rational functions, are continuous on their domain.

1. Which function is continuous on the interval?
   (a) $\dfrac{1}{\cos(x)}$ on $[0, \pi]$.
   (b) $\dfrac{\mathrm{e}^x}{\mathrm{e}^x - 1}$ on $[-1, 1]$

2. Are following functions continuous?
   (a) $f(x) = \begin{cases} x & x \leqslant 1 \\ x^2 & x > 1 \end{cases}$
   (b) $g(x) = \begin{cases} x & x \leqslant 3 \\ x^3 & x > 3 \end{cases}$

3. Let f be the function given by

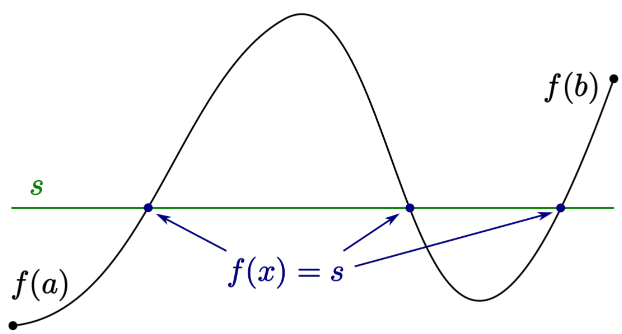
$$f(x) = \begin{cases} 4 - x & 0 \leqslant x \leqslant 3 \\ x^2 - 8x + 17 & 3 < x < 5 \\ 12 - 2x & 5 \leqslant x \leqslant 6 \end{cases}$$

   (a) Find all values of x for which f is not continuous.

   (b) List the largest open intervals on which f is continuous.

2.1 Intermediate Value Theorem

Theorem 2.2 (IVT)**.** *Suppose f is continuous on a closed interval $[a, b]$. If s is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = s$.*



3 Finding Limits

3.1 Use continuity and Algebra

If a function $f(x)$ is continuous at $x = c$, the limit is the value of $f(x)$ there, i.e.

$$\lim_{x \to c} f(x) = f(c)$$

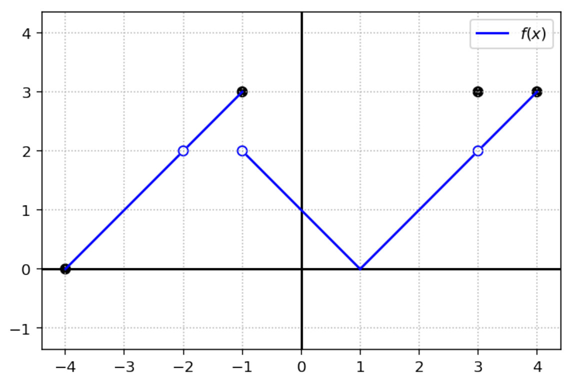
We can also use algebra to find certain limits, for example, the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not defined for

$x = 3$, we cannot plug in $x = 3$ to find the limit $\lim_{x\to 3} f(x)$. However, we can write

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x\to 3} \frac{x+3}{1} \\ &= 3+3 = 6\end{aligned}$$

1. Find the limit exactly using algebra.
   (a) $\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$
   (b) $\lim_{h\to 0} \frac{(5+h)^2-5^2}{h}$

2. Use the graph of $f(x)$ below to give estimations for each expression if they exist. Indicate if they do not exist.



(a) $\lim_{x\to -3} f(x)$
(b) $\lim_{x\to -2} f(x)$
(c) $\lim_{x\to -1} f(x)$
(d) $\lim_{x\to 0} f(x)$
(e) $\lim_{x\to 1} f(x)$
(f) $\lim_{x\to 3} f(x)$

3. Find the value of k such that

$$\lim_{x \to 2} \frac{(x+6)(x-k)}{x^2+x} = 4$$

4. Find k so that the function is continuous.

(a) $g(t) = \begin{cases} t+k & t \leqslant 5 \\ kt & t > 5 \end{cases}$

(b) $h(x) = \begin{cases} kx & 0 \leqslant x \leqslant 1 \\ 2kx+3 & 1 < x \leqslant 5 \end{cases}$

3.2 Limits do not exist

There are cases that the limit does not exist, for example

- $f(x) = \dfrac{|x-2|}{x-2}$ when $x \to 2$
- $g(x) = \dfrac{1}{x^2}$ when $x \to 0$
- $h(x) = \sin(1/x)$ when $x \to 0$