# Rational Functions

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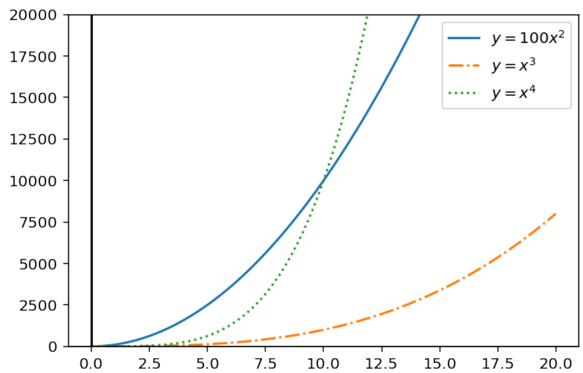
## 1 Functions

### 1.1 Power functions

A *power function* is a function of following form

$$f(x) = kx^p$$

where $k, p$ are constants.



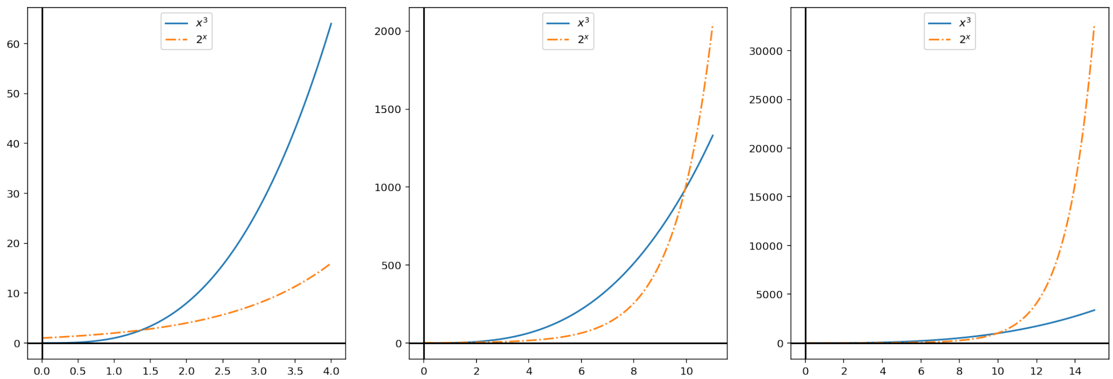

### 1.2 Dominance

**Definition 1.1.** We say that a function $f(x)$ *dominates* another function $g(x)$ if $f(x)$ eventually grows faster than $g(x)$.

For different power functions $k_1x^{n_1}$ and $k_2x^{n_2}$ with $k_1, k_2 > 0$, $k_1x^{n_1}$ dominates $k_2x^{n_2}$ whenever $n_1 > n_2$, as shown above.

Next let's discuss the dominance between power functions and exponential functions.

Look at following graphs of $x^3$ and $2^x$,




Takeaway: Every exponential growth function eventually dominates every power function.

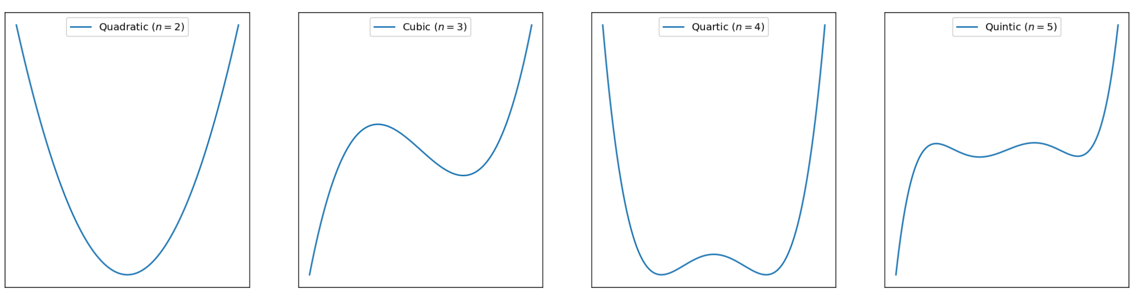
1. Which function dominates as $x \to \infty$?
   (a) $10e^{0.1x}$ or $0.2x^5$
   (b) $\sqrt{x}$ or $\ln(x)$

## 1.3 Polynomials

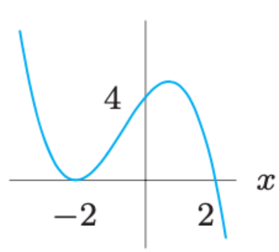
Polynomials are the sums of power functions with nonnegative integer exponents.

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

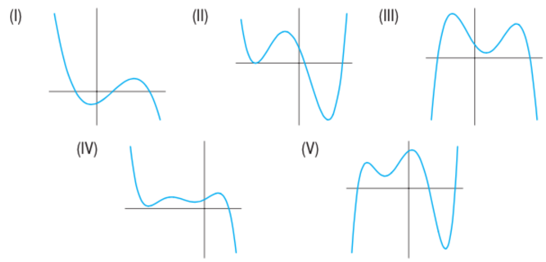
Here $n$ is an nonnegative integer called the *degree* of the polynomial. The term $a_n x^n$ is called the *leading term*, and $a_n \neq 0$ is the *leading coefficient*.




1. Find the cubic polynomial for following graph.



2. Each of the graphs below is of a polynomial. The windows are large enough to show end behavior.



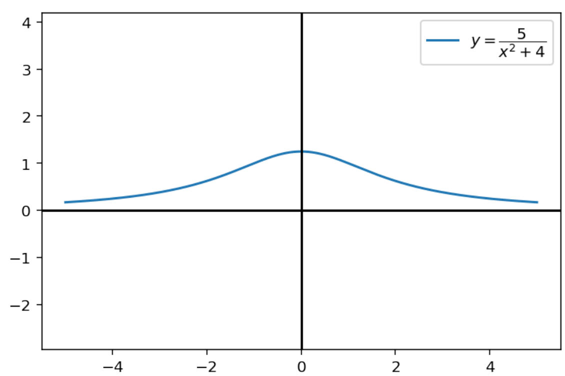

   (a) What is the minimum possible degree of the polynomial?
   (b) Is the leading coefficient of the polynomial positive or negative?

## 1.4 Rational functions

Rational functions are ratios of polynomials

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x), q(x)$ are polynomials with $q(x) \neq 0$.




What is the range of this function $\dfrac{5}{x^2+4}$?

1. A box of fixed volume $V$ has a square base with side length $x$. Write a formula for the height $h$ of the box in terms of $x$ and $V$. Sketch a graph of $h$ versus $x$.

2. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time $t$ seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?

## 2 Asymptotes

If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is called a *horizontal asymptote*. This occurs when

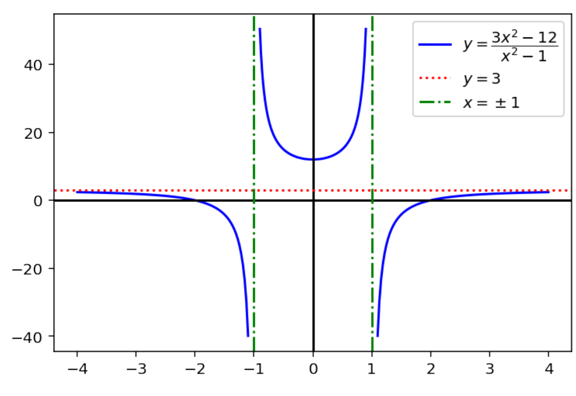
$$f(x) \to L \quad \text{as} \quad x \to \infty \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to -\infty$$

If the graph of $y = f(x)$ approaches a vertical line $x = K$ as $x \to K$ from one side or the other side, then the line $x = K$ is called a *vertical asymptote*. This occurs when

$$y \to \infty \quad \text{or} \quad y \to -\infty \quad \text{when} \quad x \to K$$

We call the behavior of a function as $x \to \pm\infty$ its *end behavior*.

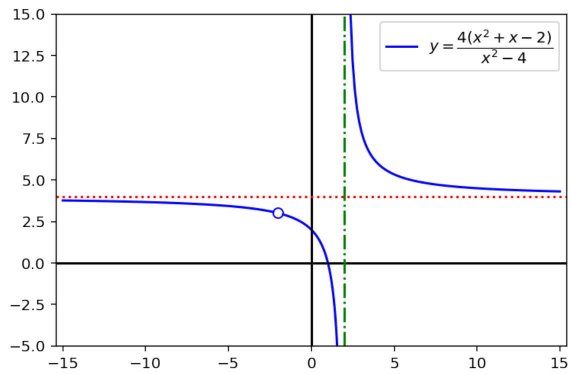
Let us look at the graph of the function $y = \dfrac{3x^2 - 12}{x^2 - 1}$.




For a rational function $\dfrac{p(x)}{q(x)}$,

- the vertical asymptotes are determined by ______________________________.
- the horizontal asymptotes are determined by ______________________________.

Let's look at another example! Below is the graph of the function $y = \dfrac{4(x^2 + x - 2)}{x^2 - 4}$




## 3 Questions

1. Determine the end behaviour of following functions
   (a) $f(x) = 1000 - 38x + 50x^2 - 5x^3$
   (b) $f(x) = \dfrac{3x^2 + 5x + 6}{x^2 - 4}$

2. Find all horizontal and vertical asymptotes for each rational function
   (a) $f(x) = \dfrac{x^2 + 5x + 4}{x^2 - 4}$
   (b) $f(x) = \dfrac{5x^3 + 7x - 1}{x^3 - 27}$

3. For each function, fill in the blanks in the statements:

- $f(x) \to$ ____________ as $x \to -\infty$,
- $f(x) \to$ ____________ as $x \to \infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$.

(b) $f(x) = \dfrac{3x^2 - 5x + 2}{2x^2 - 8}$

(c) $f(x) = e^x$

4. A rational function $f(x) = \frac{g(x)}{h(x)}$ is graphed below. If $g(x)$ and $h(x)$ are both quadratic functions, give possible formulas for $g(x)$ and $h(x)$.