Exponential Functions

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1 Exponential Functions

1.1 General Exponential Function

Definition 1.1. A general *exponential function* of t with *base* a if

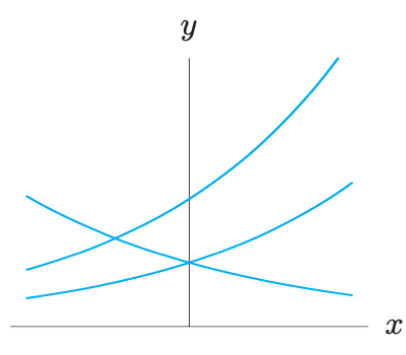
$$P(t) = P_0 a^t$$

where P_0 is the *initial quantity* (when $t = 0$).

- If $a > 1$, we have exponential growth.
- If $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leqslant 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 \cdot 1^t = P_0$ is then a constant function.

The graph of various exponential functions are shown below



Any exponential function $P(t) = P_0 a^t$ can be rewritten as an exponential function with base $\mathrm{e} = 2.718281828...$, i.e.

$$P(t) = P_0 \mathrm{e}^{kt}.$$

The relationship between a and k is following

$$\begin{aligned} \cancel{P_0} a^t &= \cancel{P_0} e^{kt} \\ a^t &= (e^k)^t \\ a &= e^k \text{ or } k = \ln(a) \end{aligned}$$

- $r = a - 1$ is called *growth rate*.
- k is called *continuous growth rate*.

QUESTIONS

1. For following functions, what is the initial quantity? What is the growth rate? State if the growth rate is continuous.
   (a) $P = 5(1.07)^t$
   (b) $P = 7.7(0.92)^t$
   (c) $P = 3.2e^{0.03t}$
   (d) $P = 15e^{-0.06t}$

2. Let $f(t) = Q_0 a^t = Q_0(1 + r)^t$. Suppose that $f(5) = 75.94$ and $f(7) = 170.86$. Find the base a and the growth rate r.

3. An air-fresher starts with 30 grams and evaporates over time. In each of the following cases, write a formula for the quantity Q in grams, of air-freshener remaining, t days after the start. The decrease is
   (a) 2 grams a day
   (b) 12% a day

4. The table gives the number of North American houses (millions) with analog cable TV.

Year	2010	2011	2012	2013	2014	2015
Houses	18.3	13	7.8	3.9	1	0.5

(a) Plot the number of houses H in millions, with cable TV versus year, Y.

(b) Could H be a linear function of Y? Why or why not?

(c) Could H be an exponential function of Y? Why or why not?

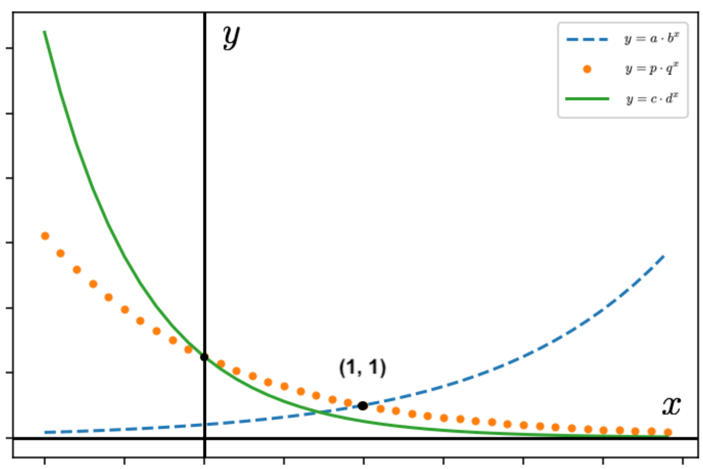
5. The exponential functions $y = a \cdot b^x, y = c \cdot d^x$ and $y = p \cdot q^x$ have b, d, q positive.

(a) Which of the constants a, c, p must be positive?

(b) Which of the constants a, b, c, d, p, q must be between 0 and 1?

(c) Which two of the constants a, b, c, d, p, q must be equal?

(d) What information about the constants a and b does the point $(1, 1)$ provide?



6. Give a possible formula for the functions graphed below.

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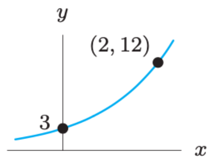

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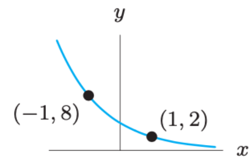

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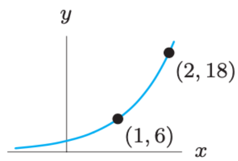

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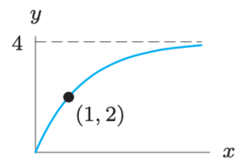
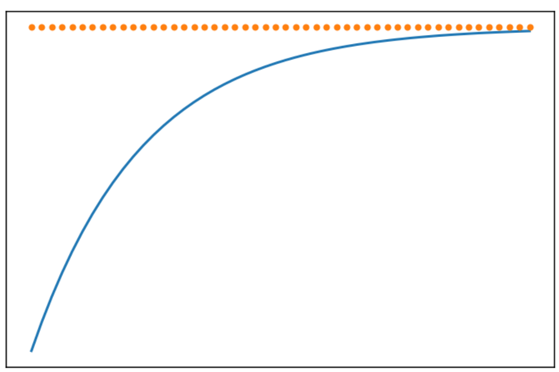


1.2 Horizontal Asymptote

If a function is of the form $y = Q_0(1 - e^{-kt})$ (or $y = Q_0(1 - a^t)$ with $0 < a < 1$), then the graph approaches $y = Q_0$ as t becomes larger and larger. The horizontal line $y = Q_0$ is called a *horizontal asymptote*.



1.3 Half-Life and Doubling Time

The *half-life* of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

The *doubling time* of an exponentially increasing quantity is the time required for the quantity to double.

If an exponential function $y = P_0 a^t$ $(0 < a < 1)$ has half-life t_0, then we know that $P_0 a^{t_0} = \frac{1}{2} P_0$. Therefore

$$\begin{aligned} P_0 a^{t_0} &= \frac{1}{2} P_0 \\ a^{t_0} &= 1/2 \\ a &= (1/2)^{1/t_0}. \end{aligned}$$

Can you derive a similar formula for the base a in terms of doubling time t_0?

QUESTIONS:

1. According to the EPA, sales of electronic devices in the US doubled between 1997 and 2009, when 438 million electronic devices sold.
   (a) Find an exponential function, $S(t)$, to model sales in millions since 1997.
   (b) What was the annual percentage growth rate between 1997 and 2009?

2 Concavity

The graph of a function is

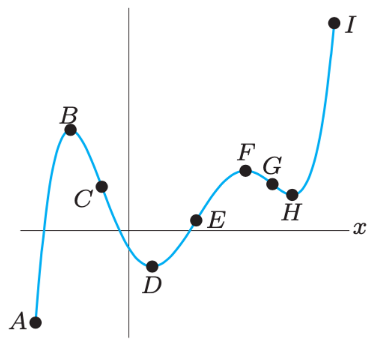
- *concave up* if it bends upward as we move left to right;
- *concave down* if it bends downward.

A line is neither concave up nor concave down.

QUESTIONS:

1. What is the concavity of the graph of an exponential function $y = P_0 a^t$? Does it depend on the value of a or P_0?

2. For which pairs of consecutive points below is the function graphed:



(a) Increasing and concave up?

(b) Increasing and concave down?

(c) Decreasing and concave up?

(d) Decreasing and concave down?