Quiz 7

Name:

2017/03/16

This quiz has 1 questions worth 10 points on 1 pages. Try to do as many questions as possible. You can use your calculator.

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1. (10 points) A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}$$

where k_1 and k_2 are positive constants which depends on the quantity of the smoke each stack is emitting. If $k_1 = 7k_2$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

Solution: The derivative is

$$\begin{aligned} \frac{dS}{dx} &= -2k_1x^{-3} - 2k_2(20-x)^{-3}(-1) \\ &= 2(k_2(20-x)^{-3} - k_1x^{-3}) \end{aligned}$$

The function is differentiable on the domain $(0, 20)$, so the only critical points are those whose derivative is zero.

Solve $\frac{dS}{dx} = 0$:

$$\begin{aligned} 2(k_2(20-x)^{-3} - k_1x^{-3}) &= 0 \\ k_2(20-x)^{-3} &= k_1x^{-3} \\ (\frac{20-x}{x})^{-3} &= \frac{k_1}{k_2} \\ \frac{20}{x} - 1 &= \sqrt[3]{\frac{k_2}{k_1}} \\ x &= \frac{20}{1+\sqrt[3]{\frac{k_2}{k_1}}} \end{aligned}$$

Since $\frac{k_2}{k_1}$, the only critical point we find is $x = \dfrac{20}{1+\sqrt[3]{\frac{1}{7}}}$.

To show that it's a local minimum, we use the second derivative test:

$$\frac{d^2S}{dx^2} = 2[-3k_2(20-x)^{-4}(-1) - (-3k_1x^{-4})]$$
$$= 6(\frac{k_2}{(20-x)^4} + \frac{k_1}{x^4}) > 0$$

The second derivative is always positive, in particular, it's positive at $x = \frac{20}{1+\sqrt[3]{\frac{1}{7}}}$. So $x = \frac{20}{1+\sqrt[3]{\frac{1}{7}}}$ is a local minimum.

Since this is the only critical point we find, it is necessarily a global minimum.