Math 115 — Practice for Exam 3

Generated March 30, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2010	3	9	plane ticket	8	
Fall 2010	3	7	coal plant	14	
Fall 2014	3	2	braking	8	
Fall 2013	3	1	excitement density	11	
Winter 2015	3	1	driving	10	
Fall 2015	3	3	race	12	
Winter 2012	3	3	truck	16	
Total				79	

Recommended time (based on points): 96 minutes

9. [8 points]
Suppose that the standard price of a round-trip plane ticket from Detroit to Paris, purchased t days after April 30, is $P(t)$ dollars. Assume that P is an invertible function (even though this is not always the case in real life).

In the context of this problem, give a practical interpretation for each of the following:

a. [2 points] $P'(2) = 55$

Solution: The standard price of a round-trip ticket from Detroit to Paris is approximately \$55 more if the ticket is purchased on May 3 than if it is purchased on May 2.

b. [2 points] $P^{-1}(690)$

Solution: The standard price of a round-trip ticket from Detroit to Paris is \$690 if it is purchased $P^{-1}(690)$ days after April 30.

c. [2 points] $\int_5^{10} P'(t)dt$

Solution: The standard price of a round-trip ticket from Detroit to Paris changes by $\int_5^{10} P'(t)dt$ dollars between May 5 and May 10. (If the integral is positive, it will be a price increase. If the integral is negative, it will be a price decrease.)

d. [2 points] $\frac{1}{5}\int_5^{10} P(t)dt$

Solution: This is the average standard price (in dollars) of a round-trip ticket from Detroit to Paris purchased between May 5 and May 10.

7. [14 points] The rate at which a coal plant releases CO_2 into the atmosphere t days after 12:00 am on Jan 1, 2010 is given by the function $E(t)$ measured in tons per day. Suppose $\int_0^{31} E(t)dt = 223$.

a. [4 points] Give a practical interpretation of $\int_{31}^{59} E(t)dt$.

Solution: $\int_{31}^{59} E(t)dt$ is the amount of CO_2 the plant releases into the atmosphere in February.

b. [4 points] Give a practical interpretation of $E(15) = 7.1$.

Solution: Since $t = 15$ corresponds to 12am on January 16th, the statement $E(15) = 7.1$ can be interpreted as
"On January 16th (or 15th) the plant releases approximately 7.1 tons of CO_2 into the atmosphere."

c. [2 points] The plant is upgrading to "clean coal" technology which will cause its July 2010 CO_2 emissions to be one fourth of its January 2010 CO_2 emissions. How much CO_2 will the coal plant release into the atmosphere in July?

Solution: Given in the problem is $\int_0^{31} E(t)dt = 223$ which means the plant released 223 tons of CO_2 into the atmosphere in January. This means that the plant will release 223/4 tons in July.

d. [4 points] Using a left-hand sum with four subdivisions, write an expression which approximates $\int_{31}^{59} E(t)dt$.

Solution: The length of the interval is 28, so with 4 subdivisions $\Delta t = 7$. This means our left hand sum is

$$\int_{31}^{59} E(t)dt \approx 7(E(31) + E(38) + E(45) + E(52)).$$

2. [8 points] A car is traveling on a long straight road. The driver suddenly realizes that there is a stop sign exactly 40 feet in front of the car and immediately hits the brakes. The car's velocity decreases for the next two seconds as the car slows to a stop.
Let $v(t)$ be the velocity of the car, in feet per second, t seconds after the driver hits the brakes. Some values of the function v are shown in the table below.

t	0	0.5	1	1.5	2
$v(t)$	40	32	23	12	0

a. [2 points] Estimate the car's acceleration 0.25 seconds after the driver hits the brakes. *Remember to show your work and include units.*

Solution: The acceleration at $t = 0.25$ can be approximated by the difference quotient

$$\frac{v(0.5) - v(0)}{0.5 - 0} = \frac{32 - 40}{0.5 - 0} = -16 \text{ ft/s}^2 .$$

Answer: -16 **ft/s^2**

b. [3 points] Based on the information in the table above, does the car first stop before, after, or at the stop sign? Or, is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Before the sign   After the sign   At the sign   [Not enough info]

Reasoning:

Solution: The distance (in feet) travelled by the car before it stops is $\int_0^2 v(t)\,dt$. Since $v(t)$ is decreasing for $0 \le t \le 2$, the left-hand sum gives an overestimate for $\int_0^2 v(t)\,dt$, while the right-hand sum gives an overestimate.

Since

$$\text{Left-hand sum} = (0.5)(40) + (0.5)(32) + (0.5)(23) + (0.5)(12) = 53.5$$

and

$$\text{Right-hand sum} = (0.5)(32) + (0.5)(23) + (0.5)(12) + (0.5)(0) = 33.5,$$

we cannot determine whether or not $\int_0^2 v(t)\,dt$ is greater than or less than 40. (The estimates 53.5 and 33.5 are the best overestimate and underestimate, respectively, that we can make of the actual distance travelled based on the data we have.)

c. [3 points] How often would speedometer readings need to be taken so that the resulting left-hand Riemann sum approximates the actual distance traveled between $t = 0$ and $t = 2$ seconds to within 1 foot?

Solution: Since $v(t)$ is decreasing, the difference between the left-hand sum and the actual value is at most the difference between the left- and right-hand sums.
When readings are taken every Δt seconds, the difference between the two sums is $(\Delta t)|v(2) - v(0)|$. We choose Δt so that $1 \ge \Delta t|v(2) - v(0))| = \Delta t(40)$. Thus, $\Delta t \le 1/40$.

Answer: Readings would need to be taken once every 1/40 **or** 0.025 seconds.

1. [11 points] At a recent UM football game, a football scientist was measuring the excitement density, $E(x)$, in cheers per foot, in a one hundred foot row of the football stadium where x is the distance in feet from the beginning of the row. He took measurements every twenty feet and the data is recorded in this table.

x	0	20	40	60	80	100
$E(x)$	30	24	19	16	13	7

Assume for this problem that $E(x)$ is a decreasing function for $0 \le x \le 100$.

a. [6 points] Write a right sum and a left sum which approximate the total cheers in the row. Be sure to write all of the terms for each sum.

Solution:
LEFT $= 20(30) + 20(24) + 20(19) + 20(16) + 20(13) = 2040$
RIGHT $= 20(24) + 20(19) + 20(16) + 20(13) + 20(7) = 1580$

b. [2 points] Indicate whether the right and left sums are overestimates or underestimates for the total number of cheers in the row.

The right sum is an **overestimate** **[underestimate]**

The left sum is an **[overestimate]** **underestimate**

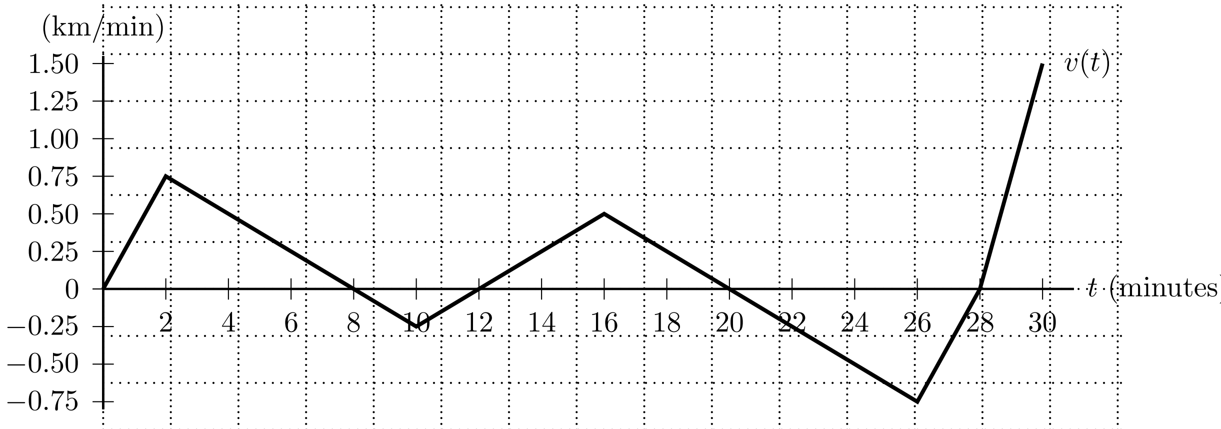
c. [3 points] How many measurements must the scientist take to guarantee that the left sum approximates the total number of cheers in the row within 5 cheers of the actual number?

Solution: The actual number of cheers is somewhere in between the left and right estimates because $E(x)$ is decreasing. So we want the difference between the left and right sums to be less than or equal to 5. If n is the number of subintervals used in the estimates, we have

$$|\text{Left} - \text{Right}| = |E(0) - E(100)| \cdot \frac{100-0}{n} = \frac{2300}{n}$$

We need $\frac{2300}{n} \le 5$, which is true if $n \ge 460$. We need 460 subintervals in our left sum estimate, so we need at least 460 measurements.

1. [10 points] Unfortunately, Sebastian left the King's castle but never made it to Adam's manor because the brakes on his car were sabotaged. Sebastian was driving on a straight road between the King's castle and Adam's manor when he found himself unable to brake and racing down a hill. Let $v(t)$ be Sebastian's velocity (in kilometers per minute) t minutes after he left the King's castle. Note that $v(t)$ is positive when Sebastian is traveling towards Adam's manor. Sebastian suspected he was being followed so he occasionally backtracked.
Sebastian crashed 30 minutes into his journey. A graph of $v(t)$ is given below.



a. [3 points] How far from the King's castle was Sebastian 12 minutes into his journey? *Include units.*

Solution: Since Sebastian initially started at the King's castle, his distance from it after 12 minutes is given by $\int_0^{12} v(t)\,dt$. To calculate this we need to calculate the signed area between the graph of $v(t)$ and the t-axis. Therefore,

$$\int_0^{12} v(t)\,dt = (0.5)(8)(0.75) - (0.5)(4)(0.25) = 2.5 \text{ km}.$$

(Note that 0.5 is the area of each box in the graph.)

Answer: **2.5 km**

b. [2 points] What was Sebastian's average velocity during the first 12 minutes of his journey?

Solution: Sebastian's average velocity during the first 12 minutes is given by the equation

$$frac112 \int_0^{12} v(t)\,dt = \frac{2.5}{12}.$$

Answer: $\frac{2.5}{12}$ **km/min**

c. [2 points] Of the four times below, circle the one at which Sebastian's acceleration was the greatest (i.e. most positive).

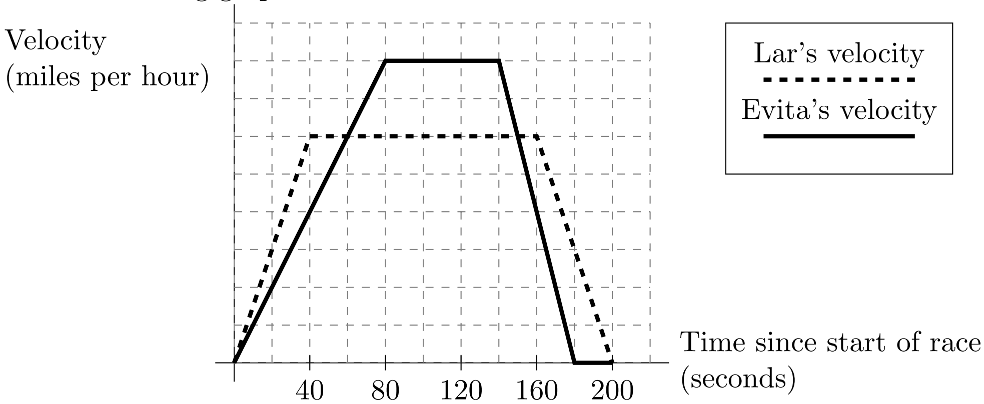
$t = 6$ $t = 13$ $t = 20$ $\boxed{t = 27}$

d. [3 points] In the interval $0 \le t \le 30$ when was Sebastian the closest to the King's castle? When was he the furthest from the King's castle?

Answer: Sebastian was the closest to the King's castle at $t =$ **0**.

Sebastian was the furthest from the King's castle at $t =$ **20**.

3. [12 points] Lar Getni and Evita Vired run a half-mile race. After the race, C.T. Latnem Adnuf receives the following graph of the two runners' velocities over the course of the race.



Unfortunately, whoever made the graph forgot to label the scale of the vertical axis, and C.T. needs your help to answer the following questions. You may assume that the horizontal grid lines are evenly spaced, but do not assume that the scales of the two axes are the same. You may also assume that both runners completed the race and then stopped running.

a. [1 point] Who won the race?

Answer: **Evita**

b. [2 points] During what time interval(s) was Lar ahead of Evita?

Answer: $0 < t < 100$

c. [2 points] During what time interval(s) was Lar running faster than Evita?

Answer: $0 < t < 60$ **and** $150 < t < 200$

d. [4 points] What was the maximum speed (in miles per hour) attained by Lar? By Evita? *Remember to show your work.*

Solution: There are 48 boxes under the graph of Lar's (and also of Evita's) velocity. Let c denote the vertical dimension of a box, in miles per hour. The horizontal dimension of a box is 20 seconds, or $\frac{20}{3600}$ hours. Since the race is $\frac{1}{2}$ a mile long, and the area under the curve is equal to the distance traveled, we must have

$$48 \cdot c \cdot \frac{20}{3600} = \frac{1}{2}$$

so $c = 1.875$. Thus, the maximum speed attained by Lar is $6 \cdot 1.875 = 11.25$ mph, and the maximum speed attained by Evita is $8 \cdot 1.875 = 15$ mph.

Answer: Lar's max speed: **11.25 mph** and Evita's max speed: **15 mph**

e. [3 points] Let $v(t)$ (respectively, $w(t)$) be Evita's (respectively, Lar's) velocity in miles per hour t seconds after the start of the race. Write an equation involving one or more integrals that expresses the following statement:

N seconds after the start of the race, Evita is M miles ahead of Lar.

Your answer may involve $v(t)$ and $w(t)$.

Answer: $\frac{1}{3600}\int_0^N (v(t) - w(t))\,dt = M$

3. [16 points] A truck is driving along a straight highway. The function $v(t)$ gives its velocity in m/sec after t seconds on the highway, and the function $a(t)$ gives its acceleration in m/sec^2 after t seconds on the highway. Consider the following table of values for $v(t)$ and $a(t)$, keeping in mind that $a(t) = v'(t)$.

t	0	10	20	30	40
$v(t)$	13	21	27	32	35
$a(t)$	1.0	0.7	0.55	0.4	0.2

In the following questions, **include units** wherever appropriate.

a. [3 points] Use a tangent line approximation to estimate the velocity of the truck after 33 seconds on the highway.

Solution:

$$v(33) \approx v(30) + 3 \cdot v'(30) = 32 + 3 \cdot 0.4 = 33.2 \text{ m/sec.}$$

b. [3 points] Do you expect that your approximation in part (a) is an overestimate or an underestimate? Briefly explain your answer based on the information in the table.

Solution: Since $a(t) = v'(t)$ is decreasing, $v(t)$ is concave down, so the approximation is an overestimate.

c. [4 points] Use a left-hand sum with four subdivisions to approximate the distance traveled by the truck in the first 40 seconds on the highway. Write out each term of the sum as well as the final answer.

Solution:

$$10(13 + 21 + 27 + 32) = 930 \text{ m.}$$

d. [3 points] Do you expect that your approximation in part (c) is an overestimate or an underestimate? Briefly explain your answer based on the information in the table.

Solution: Since $v(t)$ is increasing, the approximation is an underestimate.

e. [3 points] How frequently would velocity measurements need to be made in order to ensure that the left-hand sum and right-hand sum approximating the distance traveled in the first 40 seconds agree to within 1 meter?

Solution: We need

$$1 = (35 - 13)\Delta t,$$

so $\Delta t = 1/22$, or in other words, measurements should be made 22 times per second.